Natural vibrations of stepped arches with cracks

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Abstract. Natural vibrations of elastic circular arches are studied. The arches are assumed to be of constant width and piece wise constant height. It is assumed that at the re-entrant corners of steps stable surface cracks are located. The aim of the paper is to assess the sensitivity of the eigenfrequencies on the geometrical and physical parameters of the arch including the length and location of each crack.

Key words: elasticity, arch, natural vibrations, crack, eigenfrequency.

INTRODUCTION

The problem of vibrations of stability of beams, plates and shells have studied by several researches (see Reddy, Qatu, Vinson and Sierakowski) The natural and forced vibrations of beams weakened with the crack-like defects have been investigated by Dimarogonas (1996), Chondros et al. (1998), Kisa & Brandon (2000), Rizos at al. (1990), Nandwana & Maiti (1997) and others. Lellep & Kägo (2011, 2013) investigated the influence of cracks on eigenfrequencies of elastic stretched strips and plates.

In the previous papers by Lellep & Liyvapuu (2015 OAS, 2015 MAE) vibrations of elastic arches made of homogeneous and laminated materials were studied.

In the present paper, the results of the previous study Lellep & Liyvapuu (2015) are extended to the case of a stepped arch weakened with non-penetrated surface cracks. The cracks are assumed to be stable surface cracks. The problems of propagation of cracks are outside the scope of the present paper.

MATERIALS AND METHODS

Problem formulation

Let us study the free vibrations of a circular arch of radius $R$. It is assumed that the arch has rectangular cross section with dimensions $b$ (the width) and the total height $H$. The total height is assumed to be piece wise constant, e.g.

$$h = h_j, \quad \varphi \in (\alpha_j, \alpha_{j+1})$$

(1)

for $j = 0, \ldots, n$. 

In (1) $\varphi$ stands for the current angle (Figure 1.) and $h_0, \ldots, h_n$ and $\alpha_0, \ldots, \alpha_n$ are given constants.

Figure 1. Free supported stepped arch with crack.

Here $\alpha_0 = 0$ and $\alpha_{n+1} = \beta$.
The arch is simply supported at $\varphi = 0$ and $\varphi = \beta$.

The arch is weakened with cracks located at the re-entrant corners of steps. It is assumed that the crack located at the position $\varphi = \alpha_j$ has the length $c_j$. Evidently, the eigenfrequencies of the arch depend on the geometry of the arch and on the geometry of the crack.

The aim of the paper is to determine the eigenfrequencies of the arch and to study the sensitivity of the eigenfrequencies on the geometrical and physical parameters of the arch.

**Basic equations and assumptions**

Treating the equilibrium of an element of the vibrating arch one can conclude that (see Soedel (2004), Lellep & Liyvapuu (2015, OAS), Lellep & Liyvapuu (2015, MAE))

$$M'' + M - \bar{\rho} h_j R^2 \ddot{W} = 0$$

for $\varphi \in (\alpha_j, \alpha_{j+1})$ \quad $j = 0, \ldots, n$. Here $M$ stands for the bending moment, $W$ is the transverse displacement (deflection) and $\bar{\rho}$ is the material density. In the case of a composite or laminated material the quantity $\bar{\rho}$ is the average of densities of the layers (see Reddy (2004), Qatu (2004)).

Here and henceforth
According to the Hook’s law one has (see Lellep & Liyvapuu (2015 OAS, 2015 MAE))

\[ M = D_j \kappa = 0 \]  

(4)

for \( \varphi \in (\alpha_j, \alpha_{j+1}) \) \( j = 0, \ldots, n \). Here

\[ \kappa = -\frac{1}{R^2}(W + W'') . \]  

(5)

It is assumed herein that the axial extension \( \varepsilon = 0 \) and therefore, \( U' = -W' \).

Here \( U \) stands for the axial displacement. Note that in the case of any homogeneous material

\[ D_j = \frac{Eh_j^3}{12(1 - \nu^2)} , \]  

(6)

where \( E \) is the Young modulus and \( \nu \) – the Poisson ratio.

Assuming that both ends of the arch are simply supported one can present the boundary conditions as

\[ W(0, t) = 0; \quad M(0, t) = 0 \]  

(7)

and

\[ W(\beta, t) = 0; \quad M(\beta, t) = 0 \]  

(8)

Substituting (4) and (5) in the equilibrium equation (2) leads to the equation

\[ \frac{D_j}{R^2} (W^{IV} + 2W'' + W) + \tilde{\rho}h_j R^2 \dot{W} = 0 \]  

(9)

for \( \varphi \in (\alpha_j, \alpha_{j+1}) \) \( j = 0, \ldots, n \).

The arch under consideration has stable surface cracks at \( \varphi = \alpha_j \). It is well known that defects deteriorate the mechanical behaviour of structures. The influence of cracks on the natural vibrations of arches is modelled by the method suggested by Chondros at al. (1998) and Dimarogonas (1998). According to this method the slope of the deflection is considered as a discontinuous quantity at the cross sections with cracks. Let us denote

\[ \theta_j = W'(\alpha_j + 0, t) - W'(\alpha_j - 0, t) . \]  

(10)

It was shown in Lellep & Kägo (2013) and Lellep & Liyvapuu (2015 MAE) that on can take

\[ \theta_j = p_j \kappa(\alpha_j + 0, t) , \]  

(11)

where

\[ p_j = \frac{6\pi h_j}{1 - \nu^2} f(s_j) \]  

(12)

and

\[ t \text{ standing for time.} \]
\[ f(s_j) = 1.86s_j^2 - 3.95s_j^3 + 16.37s_j^4 - 34.23s_j^5 + 76.81s_j^6 - 126.93s_j^7 + 172s_j^8 - 143.97s_j^9 + 66.56s_j^{10}. \] (13)

**Solution of governing equations**

The equation (9) is a linear fourth order equation with partial derivatives. Making use of the method of separation of variables (see Lellep & Liyvapuu (2015 OAS, 2015 MAE), Soedel (2004)) one can look for the solution of (9) in the form

\[ W(\varphi, t) = w(\varphi) \cdot \sin(\omega t). \] (14)

In (14) the first term in the right hand side of the equality is assumed to be a function of the variable \( \varphi \). Substituting (14) in (9) leads to the ordinary differential equation of the fourth order

\[ \frac{D_j}{R^2} (w^{IV} + 2w'' + w) + \bar{\rho} h_j R^2 \omega^2 w = 0 \] (15)

for \( \varphi \in (\alpha_j, \alpha_{j+1}) \) \( j = 0, \ldots, n \).

Evidently, the general solution of (15) can be presented as

\[ w = C_{1j} \cosh(\mu_j \varphi) + C_{2j} \sinh(\mu_j \varphi) + C_{3j} \cos(\nu_j \varphi) + C_{4j} \sin(\nu_j \varphi) \] (16)

where \( C_{1j} - C_{4j} \) are arbitrary constants and

\[ \mu_j = \sqrt{1 - \omega R^2 \frac{\bar{\rho} h_j}{D_j}}, \quad \nu_j = \sqrt{1 + \omega R^2 \frac{\bar{\rho} h_j}{D_j}}. \] (17)

According to (7), (8) and (14) one can present the boundary conditions for \( w(\varphi) \) as

\[ w(0) = 0, \quad w''(\beta) = 0 \] (18)

and

\[ w(\beta) = 0, \quad w''(\beta) = 0. \] (19)

The boundary conditions (18) with (16) furnish the relations

\[ C_{10} + C_{30} = 0, \quad \mu_0^2 C_{10} - \nu_0^2 C_{30} = 0. \] (20)

It immediately follows from (20) that

\[ C_{10} = C_{30} = 0, \] (21)

provided \( \mu_0^2 + \nu_0^2 \neq 0 \).

The boundary requirements (19) lead to the equations
\[ C_{1n} \cosh(\mu_n \beta) + C_{2n} \sinh(\mu_n \beta) + C_{3n} \cos(\nu_n \beta) + C_{4n} \sin(\nu_n \beta) = 0, \]
\[ \mu_n^2 (C_{1n} \cosh(\mu_n \beta) + C_{2n} \sinh(\mu_n \beta)) - \nu_n^2 (C_{3n} \cos(\nu_n \beta) + C_{4n} \sin(\nu_n \beta)) = 0. \]  

(22)

provided \( \mu_n^2 + \nu_n^2 \neq 0 \).

The particular solution of (15) must be constructed so that in each segment the solution is given by (16) and at the boundaries the requirements (21), (23) are taken into account. Moreover, at \( \varphi = \alpha_j \) the quantities \( W, M \) and \( Q = M' \) must be continuous; the slope \( W' \) must satisfy (10) — (13). Thus, \( W, \ D(W'' + W) \) and \( D(W''' + W') \) are continuous. Here

\[ D(\alpha_j - 0) = D_j; \quad D(\alpha_j + 0) = D_{j + 1} \]  

(24)

for each \( j = 1, \ldots, n \).

The continuity conditions can be presented as

\[
\begin{align*}
C_{1j} \cosh(\mu_j \alpha_j) + C_{2j} \sinh(\mu_j \alpha_j) + C_{3j} \cos(\nu_j \alpha_j) + C_{4j} \sin(\nu_j \alpha_j) \\
= C_{1j+1} \cosh(\mu_{j+1} \alpha_j) + C_{2j+1} \sinh(\mu_{j+1} \alpha_j) \\
+ C_{3j+1} \cos(\nu_{j+1} \alpha_j) + C_{4j+1} \sin(\nu_{j+1} \alpha_j); \\
\end{align*}
\]

(25)

\[
\begin{align*}
\mu_{j+1} (C_{1j+1} \sinh(\mu_{j+1} \alpha_j) + C_{2j+1} \cosh(\mu_{j+1} \alpha_j)) \\
+ \nu_{j+1} (C_{3j+1} \sin(\nu_{j+1} \alpha_j) + C_{4j+1} \cos(\nu_{j+1} \alpha_j)) \\
= \mu_j (C_{1j} \sinh(\mu_j \alpha_j) + C_{2j} \cosh(\mu_j \alpha_j)) \\
+ \nu_j (C_{3j} \sin(\nu_j \alpha_j) + C_{4j} \cos(\nu_j \alpha_j)) \\
+ \frac{p_j D_j}{R^2} \left[ C_{1j} (1 + \mu_j^2) \cosh(\mu_j \alpha_j) + C_{2j} (1 + \mu_j^2) \sinh(\mu_j \alpha_j) \\
+ C_{3j} (1 - \nu_j^2) \cos(\nu_j \alpha_j) + C_{4j} (1 - \nu_j^2) \sin(\nu_j \alpha_j) \right]; \\
\end{align*}
\]

\[
\begin{align*}
D_j \left\{ C_{1j} (1 + \mu_j^2) \cosh(\mu_j \alpha_j) + C_{2j} (1 + \mu_j^2) \sinh(\mu_j \alpha_j) \\
+ C_{3j} (1 - \nu_j^2) \cos(\nu_j \alpha_j) + C_{4j} (1 - \nu_j^2) \sin(\nu_j \alpha_j) \right\} \\
= D_{j+1} \left\{ C_{1j+1} (1 + \mu_{j+1}^2) \cosh(\mu_{j+1} \alpha_j) \\
+ C_{2j+1} (1 + \mu_{j+1}^2) \sinh(\mu_{j+1} \alpha_j) \\
+ C_{3j+1} (1 - \nu_{j+1}^2) \cos(\nu_{j+1} \alpha_j) \\
+ C_{4j+1} (1 - \nu_{j+1}^2) \sin(\nu_{j+1} \alpha_j) \right\}; \\
\end{align*}
\]

(25)
NUMERICAL RESULTS AND DISCUSSION

The system of equations (25) augmented with (21) and (23) present a system of equation for determination of unknown $C_{ij}$ where $i = 1, \ldots, 4$ and $j = 0, \ldots, n$. This system consists of $4n + 4$ equations with the same number of unknowns. Since the system is a linear homogeneous system a non-trivial solution exists if its determinant $\Delta$ vanishes.

The solution of equation $\Delta = 0$ admits to define the eigenfrequencies. The solution procedure is implemented with the aid of the computer code MATLAB. The results of calculations are presented in Fig. 2 — Fig. 5 for the arch with a single step ($n = 1$) and $R = 1\, m$, $h_0 = 0.02\, m$, $h_1 = 0.01\, m$.

The material of the arch is a mild steel with $E = 2.1 \cdot 10^{11}\, Pa$, $\nu = 0.3$.

The influence of the first natural frequency on the location of the step is illustrated in Fig. 2 for the elastic arch with $\beta = 1$. Different curves in Fig. 2 correspond to different values of the crack depth. The uppermost curve in Fig. 2 corresponds to the arch without any defects. It can be seen from Fig. 2 that the highest values of the natural frequency are obtained in the case of arch which is free of cracks.

The natural frequency versus the step location is depicted in Fig. 3 — Fig. 5 for different values of the crack length. Different curves in Fig. 3 — Fig. 5 correspond to the arches with the central angle $\beta = 1.0$; $\beta = 1.2$; $\beta = 1.5$; $\beta = 1.6$; $\delta = 1.7$ and $\beta = 1.8$, respectively. Note that Fig. 3 is associated with the arch which has no any defect. It can be seen from Fig. 3 that the larger is the central angle of the arch, the lower is the natural frequency as might be expected. Note that similar relationship between the length and the natural frequency takes place in the case of beams, as well. In the case of beams it reads: the longer is the beam the lower is the natural frequency. Similar results are presented in Fig. 4 and Fig. 5 for arches with crack lengths $c = 0.6h_1$ and $c = 0.8h_1$ respectively.

It can be seen from Fig. 5 that the upper curves associated with $\beta = 1.0$ and $\beta = 1.2$ are decreasing in the range of small values of $\alpha$. If, however, $\alpha > 0.4$ the function $\omega = \omega(\alpha)$ are increasing everywhere. In the particular case if $h_0 = h_1$ the natural frequency $\omega$ decreases monotonically with increasing value of $\beta$ (see Lellep & Liyvapuu (2015 OAS, 2015 MAE)).
Figure 2. Natural frequency of the arch vs. depth of the crack.

Figure 3. Natural frequency versus step location (s=0).
Figure 4. Natural frequency versus step location (s=0.6).

Figure 5. Natural frequency versus step location (s=0.8).
CONCLUSIONS

Natural vibrations of circular arches with piece wise constant thickness have been considered. An analytical method for determination of eigenfrequencies of arches with cracks was developed. It was shown that the parameters of the crack essentially influence on the vibration of the arch. The highest value of the natural frequency corresponds to the arch with any defects.

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