COMPARISON OF SPEEDS OF CONVERGENCE IN SOME FAMILIES OF FUNCTIONAL SUMMABILITY METHODS

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We compare speeds of convergence in certain families of summability methods for functions.

1. Let us consider the functions \( x = x(u) \) defined for \( u \geq 0 \), bounded and measurable by Lebesgue on every finite interval \([0, u_0]\). Let us denote the set of all these functions by \( X \).

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Suppose that \( A \) is a transformation of functions \( x = x(u) \) (or, in particular, of sequences \( x = (x_n) \)) into functions \( Ax = y = y(u) \in X \). If the limit \( \lim_{u \to \infty} y(u) = s \) exists then we say that \( x \) is convergent to \( s \) with respect to the summability method \( A \), and write \( x(u) \to s(A) \).

If the function \( y = y(u) \) is bounded then we say that \( x \) is bounded with respect to the method \( A \), and write \( x(u) = O(A) \).

We denote by \( \omega_A \) the set of all these functions \( x \), where the transformation \( A \) is applied. The summability method \( A \) is said to be regular if

\[
\lim_{u \to \infty} x(u) = s \implies \lim_{u \to \infty} y(u) = s
\]

whenever \( x \in X \).

The most common summability method for functions \( x \) is an integral method \( A \) defined with the help of transformation

\[
y(u) = \int_0^\infty a(u, v) x(v) dv
\]

where \( a(u, v) \) is certain function of two variables \( u \geq 0 \) and \( v \geq 0 \).

For sequences \( x = (x_n) \) we do not consider in our paper the matrix methods (which are the most common summability methods) but focus ourselves on certain semi-continuous summability methods \( A \) defined by transformations

\[
y(u) = \sum_{n=0}^\infty a_n(u) x_n \quad (u \geq 0)
\]

where \( a_n(u) \) \((n = 0, 1, 2, \ldots)\) are some functions from \( X \).

One of the basic notions in our paper is the notion of speed of convergence. We follow here the definitions given for sequences in [4] and [5].

Let \( \lambda = \lambda(u) \) be a positive function from \( X \) such that \( \lambda(u) \to \infty \) as \( u \to \infty \). We say that a function \( x = x(u) \) is convergent to \( s \) with speed \( \lambda \) if the finite limit

\[
\lim_{u \to \infty} \lambda(u) |x(u) - s|
\]

exists. Note that the limit can be zero. If we have

\[
\lambda(u) |x(u) - s| = O(1)
\]

as \( u \to \infty \) then \( x \) is said to be bounded with speed \( \lambda \). We use the notations \( c^\lambda \) and \( m^\lambda \) for the sets of all these functions \( x = x(u) \) which are convergent to some \( s \) with speed \( \lambda \) and bounded
with speed $\lambda$, respectively. In the obvious manner the notion of speed can be transferred to summability methods. We say that $x$ is convergent or bounded with speed $\lambda$ with respect to the summability method $A$ if $Ax \in c^\lambda$ or $Ax \in m^\lambda$, respectively.

2. We discuss the Riesz-type families $\{A_\alpha\}$ of summability methods $A_\alpha$ where $\alpha > \alpha_1$ and $\alpha_1$ is some fixed number (see [6], [9]).

**Definition.** A family $\{A_\alpha\}$ ($\alpha > \alpha_1$) is said to be a Riesz-type family if for every $\beta > \gamma > \alpha_1$ the relation $\omega A_\gamma \subset \omega A_\beta$ holds and the methods $A_\gamma$ and $A_\beta$ are connected through

$$y_\beta(u) = \frac{M_{\gamma,\beta}}{r_\beta(u)} \int_0^u (u-v)^{\beta-\gamma-1} r_\gamma(v) y_\gamma(v) \, dv \quad (u > 0),$$

(1)

where $r_\gamma = r_\gamma(u)$ and $r_\beta = r_\beta(u)$ are some positive functions from $X$ related through

$$r_\beta(u) = M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1} r_\gamma(v) \, dv \quad (u > 0)$$

(2)

and $M_{\gamma,\beta}$ is a constant depending on $\gamma$ and $\beta$.

In other words, a Riesz-type family is a family where every two methods are connected through the connection formula

$$A_\beta = C_{\gamma,\beta} \circ A_\gamma \quad (\beta > \gamma > \alpha_1),$$

where $C_{\gamma,\beta}$ is the transformation defined by relations (1) and (2).

**Example 1.** Let $\{A_\alpha\}$ be the family of generalized integral Nörlund methods $(N, p_\alpha(u), q(u))$ ($\alpha > 0$) defined with the help of transformation

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \int_0^u p_\alpha(u-v) q(v) x(v) \, dv,$$

where

$$r_\alpha(u) = \int_0^u p_\alpha(u-v) q(v) \, dv > 0,$$

$$p_\alpha(u) = \int_0^u (u-v)^{\alpha-1} p(v) \, dv \quad (u > 0, \ \alpha > 0)$$

and $p = p(u) \in X$ and $q = q(u) \in X$ are some positive functions. It is known that the relations (1) together with (2) and

$$M_{\gamma,\beta} = \frac{\Gamma(\beta)}{\Gamma(\gamma) \Gamma(\beta-\gamma)}$$

hold for any $\beta > \gamma > 0$ (see [10]), and thus this family is a Riesz-type family. In particular, if $p(u) = q(u) = 1$ ($u \geq 0$) we get the Riesz methods $(R, \alpha)$ (see [3]).

**Example 2.** Consider the family $\{A_\alpha\}$ of Borel-type methods $A_\alpha = (B, \alpha, q_n)$ (see [9]). Let $(q_n)$ be a non-negative sequence with $q_0 > 0$ such that the power series $\sum q_n u^n$ has the radius of convergence $R = \infty$ and $q_n > 0$ at least for one $n \in \mathbb{N}$. Denote

$$r_\alpha(u) = \sum_{n=1}^{\infty} \frac{n! q_n u^{n+\alpha-1}}{\Gamma(n+\alpha)}$$
and define the methods \((B, \alpha, q_n)\) \((\alpha > -1/2)\) for converging sequences \(x = (x_n)\) with the help of transformation
\[
y_{\alpha}(u) = \frac{1}{r_{\alpha}(u)} \sum_{n=1}^{\infty} n! q_n u^{n+\alpha-1} x_n \quad (u > 0).
\]
The methods \((B, \alpha, q_n)\) satisfy the relations (1) and (2) with \(M_{\gamma, \beta} = 1/\Gamma(\beta - \gamma)\) (see \([9]\)). Thus \(\{A_\alpha\}\) is a Riesz-type family. In particular, if \(q_n = \frac{1}{n!}\) we get the Borel-type methods \((B, \alpha) = (B, \alpha, 1/n!)\) (see \([1], [2]\)). If, in addition, \(\alpha = 1\) then we get the Borel method \(B = (B, 1)\).

The Riesz-type family \(\{A_\alpha\}\) has the property of monotony.

Proposition. Let \(\{A_\alpha\}\) \((\alpha > (\gamma_1)\alpha_1)\) be a Riesz-type family. Then we have for functions \(x = x(u)\) and numbers \(s\) and \(\beta > \gamma > (\gamma_1)\alpha_1\) that
\[
x(u) = O(A_{\gamma}) \implies x(u) = O(A_{\beta}) \quad \text{and} \quad x(u) \to s(A_{\gamma}) \implies x(u) \to s(A_{\beta}),
\]
provided that the condition
\[
\lim_{u \to \infty} \int_0^u r_{\alpha_1}(v) \, dv = \infty
\]
is satisfied if \(\gamma = \alpha_1\) is included.

3. The following theorem was published in recent paper \([7]\).

Theorem. Let \(\{A_\alpha\}\) \((\alpha > \alpha_0)\) be a Riesz-type family. Let be given some positive function \(\lambda = \lambda(u) \to \infty\) from \(X\) and some number \(\gamma > \alpha_0\) such that \(r_{\alpha}(u) / \lambda(u) \in X\).

(i) Then we have for functions \(x = x(u)\) and numbers \(s\) and \(\beta \geq \gamma\) that
\[
\lambda(u) [y_{\gamma}(u) - s] = O(1) \implies \lambda_{\beta}(u) [y_{\beta}(u) - s] = O(1),
\]
where the speeds are related through the formulas
\[
\lambda_{\beta}(u) = \frac{r_{\beta}(u)}{b_{\beta}(u)} \quad \text{with} \quad b_{\beta}(u) = M_{\gamma, \beta} \int_0^u (u - v)^{\beta - \gamma - 1} b_{\gamma}(v) \, dv \quad \text{and} \quad b_{\gamma}(u) = \frac{r_{\gamma}(u)}{\lambda(u)}.
\]

(ii) Moreover, we have that
\[
\lambda(u) [y_{\gamma}(u) - s] \to t \implies \lambda_{\beta}(u) [y_{\beta}(u) - s] \to t,
\]
provided that
\[
\lim_{u \to \infty} \int_0^u b_{\gamma}(v) \, dv = \infty.
\]

Relations (3) and (5) can be also written as
\[
A_{\gamma} x \in m^\lambda \implies A_{\beta} x \in m^{\lambda_{\beta}}
\]
and
\[
A_{\gamma} x \in c^\lambda \implies A_{\beta} x \in c^{\lambda_{\beta}},
\]
respectively.

We remark that Theorem can be seen as an extension of Theorem 1 from \([8]\) which is given for matrix summability methods. Theorem can be applied to special Riesz-type families in order to get comparative estimations for speeds of convergence.
Example 3. Let us consider the family of Riesz methods $A_\alpha = (R, \alpha)$ ($\alpha > 0$). Let us choose the speed of convergence $\lambda(u) = (u + 1)^\rho$ ($\rho > 0$) and some number $\gamma > 0$.
Suppose that $x = x(u)$ is a function having given speed of convergence $\lambda(u)$ with respect to the method $A_\gamma = (R, \gamma)$ ($\gamma > 0$) and determine with the help of conditions (4) the speed of convergence $\lambda_\beta(u)$ of $x = x(u)$ with respect to the methods $A_\beta = (R, \beta)$ for $\beta > \gamma$. As a result, we get the following estimations:

$$
\lambda_\beta(u) \sim \frac{\Gamma(\gamma + 1) \Gamma(\beta - \rho + 1)}{\Gamma(\beta + 1) \Gamma(\gamma - \rho + 1)} \lambda(u) \quad \text{if } \rho < \gamma + 1,
$$

$$
\lambda_\beta(u) \approx \begin{cases} 
\frac{\lambda(u)}{\log u} & \text{if } \rho = \gamma + 1, \\
\lambda(u) u^{\gamma - \rho + 1} & \text{if } \rho > \gamma + 1.
\end{cases}
$$

REFERENCES