Noise-induced Transitions in a Generalized Verhulst Model with a Reflecting Boundary

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Abstract. The dynamics of a population growth model with generalized Verhulst self-regulation driven by a multiplicative three-level Markovian noise (trichotomous noise) as well as by a time-dependent deterministic stimulus is considered. In the white noise limit, the exact formulae for the conditional probability density of the population size and for the first passage time distribution are derived separately for both the reflecting and absorbing boundary conditions at the carrying capacity. In the case of multiplicative trichotomous noise, using the reflecting boundary condition, an exact analytical solution for the stationary probability distribution is found. It is established that a variation of noise parameters, such as correlation time, amplitude, and kurtosis, can cause noise-induced phase transitions. The dependence of the critical noise characteristic, which marks a transition between different phases, on other system parameters is analysed.

Keywords: Noise-induced transitions, population dynamics, generalized Verhulst model, multiplicative noise, trichotomous noise

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INTRODUCTION

Deterministic models of population growth dynamics based on a generalized Verhulst self-regulation mechanism (GVM) [1, 2] are the standard framework for a wide array of assemblies that consist of a number of elements that interact through cooperative or competitive mechanisms [3]. Several particular cases of GVM are of great use in different applications, such as the growth process of a population [4], autocatalytic chemical reactions [5], biological and biochemical systems [6], lasers physics [7], social sciences [8], grain growth in polycrystalline materials [9], etc.

In spite of these achievements, there is still a lack of complete analytical predictability. Deterministic models of population growth dynamics cannot directly handle a range of situations dominated by environmental fluctuations in the forcings or in the controlling parameters, which are intrinsic to the phenomenon [10]. In particular, several environmental systems can be described by state variables representing the availability of a resource whose dynamics is forced by climatic oscillations and diverse stochastic environmental factors.

Over the last decades, noise-induced phenomena in linear as well as in nonlinear systems with multiplicative noise have been the topic of a number of physical investigations [11]. Stochastic resonance [12, 13], noise-induced phase transitions [14, 15], memory-enhanced stability of fractional oscillators [16, 17], and stochastic transport in ratchets [11] are a few important phenomena in this field.

Although the influence of environmental multiplicative noise on population growth dynamics has been subject to intense theoretical investigations [18]-[22], those works are usually based on some particular cases of the GVM, such as simple logistic growth [18], Richards growth [21], or Gompertz law [22]. There is hardly anything available on qualitative analysis of the influence of the white or more realistic colored noises on population growth models based on the GVM introduced in [1]. In particular, a broad subclass of the GVM, which is characterized by a finite-time accessible boundary, has not been investigated in the context of stochastic population models. This is quite unjustified in view of the fact that the importance of multiplicative fluctuations and finite-time singularities (i.e., solutions that cease to be valid beyond a particular finite-time span) for stellar structures, turbulent flows, bacterial growth, etc. has been well recognized [23, 24, 25].

Thus motivated, we consider a stochastic population growth model subjected to a GVM with a finite-time accessible boundary. The influence of the variable environment is modeled by a deterministic time-dependent growth rate and by a multiplicative trichotomous noise. Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, including all cases of dichotomous noise [26, 27].

The main novel results of this paper are exact formulas for the analytical treatment of colored-noise-induced phase transitions in the systems with GVM. To avoid misunderstanding, let us mention that we use the term noise-induced phase transitions in the sense of Refs. [14, 15], meaning that this corresponds to a qualitative change in the probability
law characterizing the random variable, in response to noise parameters. The number and position of extrema of the stationary probability density are, from this point of view, merely a practical way to monitor such a qualitative change. Moreover, in the case of the white noise limit, we will find the exact expressions for the transition probability densities as well as for the first passage time probability density, to characterize the population growth process in a more comprehensive way.

The structure of the paper is as follows. In the first section we present the model investigated in this work. A master equation description of the model is given and the externally applied boundary conditions are formulated. In the second section exact formulas are found for the transition probability densities and first passage time distribution to describe the behavior of the system in the white noise limit. The third section analyzes the behavior of the stationary state of the system. The phenomenon of noise-induced phase transitions versus the trichotomous noise parameters is established, and the critical noise characteristic is found which marks the transitions between different phases. The last section contains some brief concluding remarks.

**MODEL**

Our starting point is the stochastic differential equation for a generalized Verhulst model [2], driven by a multiplicative noise

\[
dX(t) = \left[ r(t) + \sqrt{2r(t)} Z(t) \right] X^{1+\alpha} \left( 1 - (\gamma X)^\beta \right)^\delta dt,
\]

where \( X(t) \) is the population size at time \( t \), \( r(t) > 0 \) is the purely time-dependent growth rate, \( 1/\gamma \) is a limiting factor on the population growth (the carrying capacity), which is imposed by environmental factors, basically food and space limitations. The exponents \( \alpha, \delta, \) and \( \beta \), with \( \delta, \beta, \alpha > 0 \), are free parameters, which are determined by the properties of the particular system considered.

Regarding the random function \( Z(t) \), we assume it to be a zero-mean trichotomous Markovian stochastic process [26, 27] which consists of jumps among three values \( z = \{a, 0, -a\} \). The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities \( p_0(a) = p_0(-a) = q, \) and \( p_0(0) = 1 - 2q \). In a stationary state, the fluctuation process satisfies

\[
\langle Z(t) \rangle = 0, \quad \langle Z(t + \tau) Z(t) \rangle = 2qa^2 e^{-\nu \tau},
\]

where the switching rate \( \nu \) is the reciprocal of the noise correlation time \( \tau_0 = 1/\nu \). It is remarkable that the kurtosis of the trichotomous noise, \( \kappa = (1/2q) - 3 \), can be anything from \(-2\) to \( \infty \). The hypothesis that the influence of the noise on the growth rate is proportional to \( \sqrt{r(t)} \) in Eq. (1) is a reasonable assumption for many systems given that the stochastic variability of the process is expected to increase during an increasing growth rate season (cf. also [28]).

Let us now consider the deterministic counterpart of Eq. (1) with \( r = \text{const} > 0 \), and \( Z(t) = 0 \). If the initial population size \( X(0) = x_0 \) lies within the interval \((0, 1/\gamma)\), then the solution \( X(t) \) remains in this interval, \( 0 < X(t) < 1/\gamma \), for all finite values of the time \( t \geq 0 \). In the deterministic case the behavior of the population size \( X \) in time depends crucially on the values of the exponent \( \delta \). If \( \delta > 1 \), then the boundary \( x = 1/\gamma \) is not accessible in finite time, but when \( \delta < 1 \), the boundary \( 1/\gamma \) is accessible. More precisely, at a finite time \( t_c \)

\[
t_c = \frac{1}{\nu} g(x_0),
\]

where \( g(x) = \frac{\gamma^\alpha [1 - (\gamma x)^\beta]^{1-\delta}}{\beta (1 - \delta)} 2F_1 \left( 1 + \frac{\alpha}{\beta}, 1 - \delta; 2 - \delta; (1 - (\gamma x)^\beta) \right), \quad \delta < 1,
\]

and \( 2F_1 \) is the hypergeometric function, the population size \( x \) grows up to the boundary value \( 1/\gamma \), which is the fixed point for the system (1). It should be noted that systems with finite-time singularities are encountered in modeling stellar structures, turbulent flows, and bacterial growth [23, 24, 25] as well as in econophysics, geophysics, material physics, and demography [29]-[33]. In the rest of this paper, we shall always assume that the exponent \( \delta \) is restricted to the inequality

\[
0 < \delta < 1.
\]

The model described by Eq. (1) with the condition (5) can be reduced to an elementary stochastic equation by means of the transformation

\[
y = g(x),
\]
which leads to the equation
\[ \frac{dy}{dt} = -r(t) - \sqrt{2r(t)} Z(t), \quad y \in (0, \infty). \] (7)

We emphasize that for the points \( x = 0 \) and \( x = 1/\gamma \), the transformation as given by the formulae (4) and (6) does not hold. The behavior of the system in these points has to be analyzed through a limit procedure.

Following Ref. [26], we can write the master equation corresponding to Eq. (7) for the joint probability densities \( P_n(y,t) \) for the position variable \( y(t) \) and the fluctuation variable \( Z(t) \) as
\[ \frac{\partial}{\partial t} P_n(y,t) = \left[ r(t) + \sqrt{2r(t)} \right] \frac{\partial}{\partial y} P_n(y,t) - \nu \left[ P_n(y,t) - P_3(z_n) \sum_{m=1}^{3} P_m(y,t) \right], \quad n = 1, 2, 3. \] (8)

The probability density \( P(y,t) \) in the \( y \) space and the probability current \( J(y,t) \) is then evaluated via the joint probability densities \( P_n(y,t) \)
\[ P(y,t) = \sum_{n=1}^{3} P_n(y,t), \quad J(y,t) = -r(t)P(y,t) - a\sqrt{2r(t)} \left[ P_1(y,t) - P_3(y,t) \right]. \] (9)

In the limit \( \nu \to \infty, a \to \infty \), so that
\[ \lim_{\nu \to \infty, a \to \infty} \frac{4qa^2}{\nu} = D \] (10)

is finite, the trichotomous noise \( Z(t) \) reduces to a \( \delta \)-correlated white noise with the intensity \( D \). The corresponding Fokker-Planck equation reads
\[ \frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y} J(y,t), \] (11)

where the probability current is expressed as
\[ J(y,t) = -r(t)P(y,t) - D r(t) \frac{\partial}{\partial y} P(y,t). \] (12)

For further analysis of the process described by Eq. (1) we should specify a boundary condition at \( x = 1/\gamma \) (or \( y = 0 \)). Although the model (1) with the condition (5) without supplementary assumptions predicts the absorbing boundary condition at \( x = 1/\gamma \), for the modeling of some systems it is, however reasonable to consider Eq. (1) with a reflecting boundary at \( x = 1/\gamma \) (e.g., some biological populations including the human population, cf. also [29, 30]). Hence, in the rest of this paper, we assume at \( y = 0 \) the boundary condition
\[ \rho P(0,t) + (1 - \rho)J(0,t) = 0, \] (13)

where \( 0 \leq \rho \leq 1 \) and \( 1 - \rho \) are the probabilities that the boundary \( y = 0 \) is absorbing or reflecting, respectively. Due to the linearity of Eqs. (8) and (11), the solution in the presence of the condition (13) is a superposition of the solutions \( P^{(a)}(y,t) \) and \( P^{(r)}(y,t) \), which correspond to pure absorbing and reflecting boundaries, respectively, i.e.,
\[ P(y,t) = \rho P^{(a)}(y,t) + (1 - \rho)P^{(r)}(y,t). \] (14)

**MULTIPLICATIVE WHITE NOISE**

In this section we consider the noise \( Z(t) \) in Eq. (1) as a white noise with the correlation function
\[ < Z(t)Z(t') > = D\delta(|t-t'|). \] (15)

Returning to the Fokker-Planck equation (11) with the boundary condition (13) we can present exact expressions for the conditional probability density \( P(y,t \mid y_0,0) \) with the initial condition
\[ P(y,0 \mid y_0,0) = \delta(y-y_0) \] (16)

and for the first passage probability density \( W(t \mid y_0) \).
Transition probability densities

Consider Eq. (11) with the conditions (13) and (16). This problem can be easily solved using the method of images [34]. In the case of an absorbing boundary ($\rho = 1$ in Eq. (13)) the barrier at $y = 0$ is replaced by a mirror source located at a point such that the solutions of the Fokker-Planck equation emanating from the original and mirror sources exactly compensate each other at the position of the barrier at each instant of time [28, 34]. Assuming the integrability of $r(t)$ this procedure gives

$$P^a(y, t \mid y_0, 0) = \frac{1}{2 \sqrt{\pi D \eta(t)}} \left\{ \exp \left[ \frac{-(y-y_0+\eta(t))^2}{4D\eta(t)} \right] - \exp \left(\frac{y_0}{D}\right) \exp \left[ \frac{-(y+y_0+\eta(t))^2}{4D\eta(t)} \right] \right\}, \quad (17)$$

where

$$\eta(t) := \int_0^t r(t') dt'. \quad (18)$$

Similarly, the solution of Eq. (11) with a reflecting boundary at $y = 0$ ($\rho = 0$ in Eq. (13)) can be obtained by the method of images. The solution then becomes (cf. also [28])

$$P^r(y, t \mid y_0, 0) = \frac{1}{2 \sqrt{\pi D \eta(t)}} \left\{ \exp \left[ \frac{-(y-y_0+\eta(t))^2}{4D\eta(t)} \right] + \exp \left(\frac{y_0}{D}\right) \exp \left[ \frac{-(y+y_0+\eta(t))^2}{4D\eta(t)} \right] \right\} + \frac{1}{2D} \exp \left(\frac{-y}{D}\right) \text{Erfc} \left[ \frac{y+y_0-\eta(t)}{2\sqrt{D\eta(t)}} \right]. \quad (19)$$

Taking into account the transformation (6) with Eq. (4), it is easily established that for the original variable $x$ the conditional probability density $P(x, t \mid x_0, 0)$ can be written as

$$P(x, t \mid x_0, 0) = \frac{1}{x^{1+\alpha} (1 - (\gamma/x)^\alpha)^{\beta/\alpha}} P(g(x), t \mid g(x_0), 0), \quad x, x_0 \in (0, 1/\gamma). \quad (20)$$

First passage time distribution

For a process commencing at a generic position $y_0$ at $t = 0$, the time at which this process reaches the boundary $y = 0$ for the first time (first passage time) is itself a random variable whose statistics are fundamental in many branches of science such as chemistry, neural sciences, and econometrics [28]. The probability density $W(t \mid x_0)$ of the time required to reach the boundary $y = 0$ for the first time can be written in terms of the conditional probability $P^a(y, t \mid y_0, 0)$ as [34]

$$W(t \mid x_0) = D r(t) \frac{\partial}{\partial y} P^a(y, t \mid y_0, 0)|_{y=0}. \quad (21)$$

Now, from Eqs. (17) and (21) it follows that

$$W(t \mid x_0) = \frac{r(t)y_0}{2\sqrt{\pi D(\eta(t))^3/2}} \exp \left[ \frac{(\eta(t)-y_0)^2}{4D\eta(t)} \right], \quad y_0 = g(x_0). \quad (22)$$

It is remarkable that in the case of a constant growth rate, $r = \text{const}$, the mean first passage time

$$\langle T \rangle = \int_0^\infty t W(t \mid x_0) dt = \frac{1}{r} g(x_0) \quad (23)$$

is exactly the same as the critical time $t_c$ in the case of the deterministic model (see Eq. (3)). It is important to note that formulas analogous to Eq. (22) for first passage time distribution were previously considered in the context of modeling the evolution of human population [30] as well as in the modeling of water resources availability in snowmelt dominated regions [28].
In Figure 1, the case of a periodic growth rate is illustrated. Now, we assume that \( r(t) \) is in the form
\[
r(t) = r_0 (1 + \varepsilon \cos \omega t), \quad 0 \leq \varepsilon < 1, \quad r_0 > 0.
\]
(24)
The oscillatory term in Eq. (24) mimics the circumstance that several populations can be described by a growth rate that depends on the availability of a resource whose dynamics is forced by diverse environmental factors and climatic oscillations. The growth rate in the form of (24) is also investigated in neuron dynamics [35].

The dependence of the conditional probability \( \tilde{P}(x, t \mid x_0, 0) \) on the population size \( x \) at three different values of time \( t \) is illustrated in Figure 1(a). It is seen that at intermediate values of time the functional dependence of \( \tilde{P} \) on \( x \) is quite complicated (dashed line in Figure 1). At large values of time the function \( \eta(t) \) in Eqs. (17) and (19) tends to \( r_0 t \) and the conditional probability becomes a stationary form independent of the forcing frequency \( \omega \). Panel (b) of Figure 1 demonstrates the behavior of the first passage density \( W(t \mid x_0) \) at several values of the forcing frequency \( \omega \). As highlighted in Figure 1(b), the probability density \( W(t \mid x_0) \) is heavily skewed with a power law tail \( t^{-3/2} \). Due to the asymmetry of \( W(t \mid x_0) \) in time, the prospective passage time \( t_p \) differs significantly from the mean first passage time \( \langle T \rangle \) and is always shorter than \( \langle T \rangle \).

![Figure 1](image_url)

**FIGURE 1.** The behaviors of the conditional probability \( \tilde{P}(x, t \mid x_0, 0) \) and the first passage density \( W(t \mid x_0) \) in the case of a periodic growth rate (Eq. (24)), computed from Eqs. (20) and (22), respectively. The system parameter values: \( \gamma = 10^{-2}, \delta = 0.5, \beta = 2, \alpha = 1, x_0 = 20, \varepsilon = 0.8, r_0 = 1, \) and \( D = 0.1 \). Panel (a): the dependence of the conditional probability on the population size \( x \) at \( \rho = 0.5 \), and \( \omega = 1 \). Solid line, \( t = 0.0004 \); dashed line, \( t = 0.005 \); dotted line, \( t = 15 \). Panel (b): The first passage density versus time \( t \). Solid line, \( \omega = 1 \); dashed line, \( \omega = 150 \); dotted line, \( \omega = 300 \)

**THE STATIONARY CASE**

Unfortunately, in the general case of the model (1) with a trichotomous noise exact solutions, like Eqs. (17)-(22), are unknown. However, if the growth rate \( r(t) \) is such that
\[
r(t) \sim r_0, \quad r_0 = \text{const}, \quad t \to \infty,
\]
(25)
an exact formula for the stationary probability density \( P^*(x) \) can be found from Eqs. (6) and (8) with the reflecting boundary condition, Eq. (13) with \( \rho = 0 \). In this case the stationary probability density \( P^*(x) \) is given by
\[
P^*(x) = \frac{1}{D(v, a, q)} \left( \frac{1}{x^{1+a} \left( 1 - (\gamma x)^\beta \right)^a} \right) \exp \left[ -\frac{g(x)}{D(v, a, q)} \right], \quad x \in (0, 1/\gamma),
\]
(26)
where \( g(x) \) is determined by Eq. (4) and
\[
D(v, a, q) := \frac{r_0 \left( 2a^2 - r_0 \right)}{v \left( r_0 - 2qa^2 + \sqrt{(2qa^2 - r_0)^2 + r_0 (2a^2 - r_0)} \right)}, \quad a > \sqrt{\frac{r_0}{2}}.
\]
(27)
Here we emphasize that the formula (26) is applicable only for sufficiently large values of the amplitude of the trichotomous noise $Z(t), a > \sqrt{\nu_0/2}$. If $a < \sqrt{\nu_0/2}$, then the normalized stationary probability density is absent. This is an indication that the population size $x$ concentrates at the boundary $x = 1/\gamma$ as the time $t$ tends to infinity. In the white noise limit (see Eq. (10)) the quantity $\dot{D}$ reduces to the intensity $D$ of the white noise, i.e., $\dot{D} = D$, and thus the formula (26) is applicable also in this case (c.f. also Eqs. (19) and (20) at $t \to \infty$).

In Figure 2, the probability density $P^s(x)$ is plotted as a function of the population size $x$ for various values of the noise parameters $\dot{D}$ and $\nu$. An important observation here is that two types of curves are distinguishable in Figure 2: namely, the monotonically increasing ones and the ones with a local maximum and minimum. As the noise parameter $\dot{D}$ increases, $P^s(x)$ undergoes a phase transition from a state without local extrema to one with two extrema.

![FIGURE 2. The stationary probability $P^s(x)$ versus the population size $x$, computed from Eqs. (26) and (4). The system parameter values: $\gamma = 10^{-2}, \delta = 0.6, \beta = 1.5, \alpha = 0.5$. Panel (a): the probability $P^s(x)$ at various values of the noise intensity $\dot{D}$. Solid line, $\dot{D} = 0.1(\dot{D} < \dot{D}_c)$; dashed line, $\dot{D} = 0.15(\dot{D} = \dot{D}_c)$; dotted line, $\dot{D} = 0.35(\dot{D} > \dot{D}_c)$. Panel (b): the probability $P^s(x)$ for several values of the noise correlation time $\tau_0 = 1/\nu$. The noise intensity $\dot{D}$ is computed from Eq. (27) at $\nu_0 = 1, a = 2, and q = 0.3$. Solid line, $\tau_0 = 0.05$; dashed line, $\tau_0 = 0.1$; dotted line, $\tau_0 = 0.15$.](image)

From Eq. (27) it can be easily seen that $\dot{D}$ increases monotonically from zero to infinity as the noise correlation time $\tau_0 = 1/\nu$ increases from 0 to $\infty$ or as the noise amplitude increases. Thus a variation of noise parameters causes a qualitative change in the stationary distribution of the population size, i.e., a phenomenon called noise-induced phase transition occurs. The critical value $\dot{D}_c$ of the noise characteristic $\dot{D}(\nu,a,q)$ can be obtained by separating the two phases from each other. Equation (26) shows that the critical parameter $\dot{D}_c$ is given by the system of equations

$$\frac{d}{dx}P^s(x) = \frac{d^2}{dx^2}P^s(x) = 0, \quad \dot{D} = \dot{D}_c. \tag{28}$$

Now, from Eqs. (26) and (28) we can easily obtain an expression for $\dot{D}_c$,

$$\dot{D}_c = \frac{\gamma^a (1 - \sigma)^{1 - \delta}}{\sigma^\beta [1 + \alpha - (1 + \alpha + \delta \beta) \sigma]}, \tag{29}$$

where

$$\sigma = \frac{1}{2(\alpha + \beta \delta)(1 + \alpha + \beta \delta)} \left\{ 2\alpha(1 + \alpha + \beta \delta) + \beta \delta (1 + \beta) - \beta \sqrt{4 \alpha(1 + \alpha + \beta \delta) + \delta (1 + \beta)^2} \right\}. \tag{30}$$

Thus, in the phase space of the noise parameters $\nu,a,q$ the surface

$$\dot{D}(\nu,a,q) = \dot{D}_c \tag{31}$$

depicts the noise-induced phase transitions between two qualitatively different phases of the population distribution. Another important observation, apparent from Figure 2(b), is that the peak which corresponds to the local maximum of $P^s(x)$ gets more and more pronounced and shifts to smaller values of the population size as $\dot{D}$ increases. Therefore, in
the stationary regime the probability to find populations with relatively small sizes increases when the noise parameter \( \tilde{D} \) grows.

To illustrate the dependence of the mean population size \( \langle X \rangle \) and the coefficient of variation

\[
\Lambda := \sqrt{\frac{\langle X^2 \rangle - \langle X \rangle^2}{\langle X \rangle}}
\]

on noise parameters we restrict ourselves to the particular case

\[
\delta = \frac{1}{2}, \quad \beta = 2 \alpha, \quad \alpha > 1.
\]

Using the expression (26) we obtain

\[
\langle X \rangle = \frac{\sqrt{\pi}}{\gamma} \left( \frac{\gamma^a}{2\tilde{D}} \right)^{\frac{1}{2} + \frac{\alpha}{2}} \Gamma \left( 1 - \frac{1}{2\alpha} \right) \left\{ \frac{1}{\Gamma^2(1 - \frac{1}{2\alpha})} \left[ H_{\frac{1}{2}(1 - \frac{1}{\alpha})} \left( \frac{\gamma^a}{D} \right) - \frac{1}{\Gamma(1 - \frac{1}{2\alpha})} \left( \frac{\gamma^a}{D} \right)^2 \right] - 1 \right\}^{\frac{1}{2}},
\]

\[
\Lambda = \frac{\sqrt{2\tilde{D}}}{\pi \gamma^a} \left\{ \frac{1}{\Gamma^2(1 - \frac{1}{2\alpha})} \left[ H_{\frac{1}{2}(1 - \frac{1}{\alpha})} \left( \frac{\gamma^a}{D} \right) - \frac{1}{\Gamma(1 - \frac{1}{2\alpha})} \left( \frac{\gamma^a}{D} \right)^2 \right] - 1 \right\}^{\frac{1}{2}},
\]

where \( \Gamma(y) \) is the gamma function, and \( H_n(y) \) and \( N_n(y) \) are the Struve and Neumann functions, respectively. Figure 3 shows typical dependencies of \( \langle X \rangle \) and \( \Lambda \) on the noise parameter \( \tilde{D} \) at various values of the exponent \( \alpha \).

**FIGURE 3.** The stationary mean value of the population size \( \langle X \rangle \) and the coefficient of variation \( \Lambda \) versus the noise intensity \( \tilde{D} \) (see Eqs. (34) and (35)) at \( \gamma = 10^{-2} \). The solid, dashed, and dotted lines correspond to values of the system parameter \( \alpha = 1.2, \alpha = 1.8, \) and \( \alpha = 2.5, \) respectively.

From Eqs. (34) and (35) it follows that in the limit of low noise, \( \tilde{D} \to 0, \) the mean population abundance tends to the carrying capacity \( 1/\gamma, \) while the coefficient of variation \( \Lambda \) tends to zero. In the case of strong noise, \( \tilde{D} \to \infty, \) the mean population size \( \langle X \rangle \) decays asymptotically as

\[
\langle X \rangle \sim \Gamma \left( 1 - \frac{1}{\alpha} \right) \tilde{D}^{-\frac{1}{2}}.
\]

The behavior of \( \Lambda \) is more supple. If the exponent \( \alpha \) is in the interval \( 1 < \alpha < 2, \) then by increasing \( \tilde{D} \) the coefficient of variation \( \Lambda \) increases unrestrictedly, but when \( \alpha > 2, \) the coefficient of variation tends to the finite value

\[
\Lambda = \sqrt{\frac{\Gamma(1 - \frac{2}{\alpha})}{\Gamma^2(1 - \frac{1}{\alpha})}} - 1, \quad \tilde{D} \to \infty.
\]

It is interesting to note that in the limit of strong noise, \( \tilde{D} \to \infty, \) the asymptotic behavior of \( \langle X \rangle \) and \( \Lambda \) is independent of the carrying capacity \( 1/\gamma. \)
CONCLUSION

In this paper we have addressed the problem of the influence of a multiplicative noise on the dynamics of population size $X$ in the presence of a generalized Verhulst self-regulation mechanism. The effect of time-variable environment on the dynamics of the population is taken into account as a variable growth rate in the model including a time-dependent deterministic part and a three-level Markovian noise. The modeling of such systems requires solving the associated master equation with some boundary conditions at the accessible boundary of the deterministic counterpart of the system (i.e., at $x = 1/\gamma$, where $\gamma$ is the reciprocal of the carrying capacity).

In the case of multiplicative white noise the exact expressions for transitional probabilities and for first passage time densities, using both the absorbing and reflecting boundaries, are derived.

Our major result is the establishment of noise-induced phase transitions between two qualitatively different forms of a stationary population size distribution. At small noise intensities the stationary distribution is characterized with one most probable state at $x = 1/\gamma$, but if the noise intensity is greater than the critical value, the distribution includes two most probable values, one at the carrying capacity and another at moderate or small values of the population size. Therefore, although in the model considered the extinction of the population is impossible, for sufficiently strong noise intensities the population size distribution is, however concentrated in the vicinity of $x = 0$ and extinction of the population may occur in nature due to the discrete structure of actual populations [3]. Moreover, the critical values of trichotomous noise parameters, such as kurtosis, amplitude, and correlation time, which mark noise-induced phase transitions, are also found. Particularly, a variation of noise correlation time can cause a transition from one phase to the other. The corresponding formulas Eqs. (27) and (29)-(31), which determine the dependence the critical values of colored noise parameters on the self-regulation parameters $\alpha$, $\beta$, and $\delta$ (see Eq. (1)) of the GVM are significant novel results of this work.

Finally, we believe that the model and the results discussed here are also of interest in fields where some particular cases of the GVM with a finite-time singularity are relevant in system modeling, e.g., in human population dynamics [29, 30].

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