



## A practical method for constructing a reflectionless potential with a given energy spectrum

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**Abstract.** A fully algebraic approach to constructing one-dimensional reflectionless potentials with any number ( $N$ ) of bound states is described. A simple and easily applicable general formula is derived, using the methods of the theory of determinants. In particular, useful properties of special determinants – the alternants – have been exploited. The modified determinant that uniquely fixes the potential contains only  $2^{N-1}$  terms, which is a huge win compared to the  $N!$  terms of the original expansion. Moreover, the modified determinant can be very easily evaluated using the properties of alternants. To this end, two useful theorems have been proved. The main formula takes an especially simple form if one aims to reconstruct a symmetric reflectionless potential. Several examples are presented to illustrate the efficiency of the method.

**Key words:** Inverse scattering, Gelfand-Levitan-Marchenko equation, reflectionless potentials, alternants, elementary symmetric polynomials.

### 1. INTRODUCTION

Formulation and solution of inverse problems is an increasingly important field of scientific research. However, compared with a well-posed (in Hadamard's sense) direct or forward problem, the corresponding inverse problem is much more difficult and, as a rule, ill-posed. Inverse scattering problem can be considered an exception to this rule. Namely, in the simplest one-dimensional case, the inverse scattering theory provides strict mathematical criteria for the existence, uniqueness and stability of the solution. It means that in this particular case the inverse problem is well-posed as well.

The related forward problem is the solution of the simplest time-independent Schrödinger equation

$$\Psi''(x) = \frac{V(x) - E}{C} \Psi(x), \quad C \equiv \frac{\hbar^2}{2m} \quad (1)$$

for a given potential  $V(x)$ , and subjected to appropriate physical boundary conditions. Eq. (1) can be easily solved numerically and thus, in principle, all spectral characteristics of the potential  $V(x)$  can be accurately ascertained.

The inverse problem is to determine the **unknown** potential starting from the known spectral characteristics. This is a serious task even in this simple case for the following reasons:

1. It is not obvious what kind of input information is actually needed to solve the problem uniquely.
2. There must be a theoretical basis (a fundamental equation) which enables to solve the problem.

3. Apart from the theoretical difficulties, another important question arises: how to obtain the necessary input data?
4. Even if the mentioned principal barriers could be overcome, the computational-technical solution of the problem is not at all trivial.

The problems 1 and 2 have been successfully solved in early 1950s for the class of potentials on the half line:  $x \in [0, \infty)$ . The necessary and sufficient conditions for the unique solution of the inverse problem have been formulated in a series of outstanding theoretical works by Marchenko [1, 2], Gelfand, Levitan [3], Krein [4, 5] and others (see, e.g., [6], section III.7 for an overview). In addition, three different methods to solve the problem have been worked out, based on the integral equations by Gelfand-Levitan [3], Marchenko [2] and Krein [5].

Solution of the inverse scattering problem on the full line ( $-\infty < x < \infty$ ) is a more challenging problem, which was first addressed by Kay [7], Moses [8], and few years later in a series of papers by Faddeev [9, 10, 11, 12]. These fundamental studies provide full description of the solution procedure, while the correct necessary and sufficient criteria for the uniqueness of the solution were given by Marchenko (see [13], section III.5). These criteria apply to the following pair of integral equations [8, 12, 13]:

$$K_1(x, y) + A_1(x + y) + \int_x^\infty A_1(z + y)K_1(x, z)dz = 0, \quad (2)$$

$$K_2(x, y) + A_2(x + y) + \int_{-\infty}^x A_2(z + y)K_2(x, z)dz = 0, \quad (3)$$

which are formally very similar to the Marchenko equation on the half line (see [13], p. 218). On the other hand, their operator-theoretical content is closer to the Gelfand-Levitan approach. Therefore, as a kind of compromise, Eqs. (2)-(3) are often called Gelfand-Levitan-Marchenko (GLM) equations. The kernel  $A_1$  in Eq. (2) (where  $y > x$ ) is completely specified by the so-called right scattering data, while the kernel  $A_2$  in Eq. (3) ( $y < x$ ) is determined by the left scattering data. These two sets of input data are equivalent: the left data are uniquely determined by the right ones and *vice versa*. In the following analysis we will rely on Eq. (2). If one is able to solve this integral equation then the potential is given by

$$V(x) = -2C \frac{dK_1(x, x)}{dx}. \quad (4)$$

Now, let us briefly discuss Problem 3 in the above list. Unfortunately, the criteria for the uniqueness of the solution are so strict that it is nearly impossible to get all the necessary input data experimentally. However, if one sets an additional constraint that the resulting potential must be reflectionless then the inverse scattering problem can be solved much more easily. Moreover, a **symmetric** reflectionless potential is uniquely determined if its full spectrum of bound states is known. In the following analysis it is assumed that the potential  $V(x)$  that corresponds to Eq. (4) is reflectionless by definition.

As the general principles of building confining reflectionless potentials are long known [12, 14], a natural question arises: is there any need to revisit the topic? A motivation comes from Problem 4 stated above: if the total number of bound states is large then the technical side of the procedure becomes important. This in turn motivates development of more efficient algorithms. In this paper, an easily applicable analytic algorithm is derived, which enables to calculate a reflectionless potential with an arbitrary number ( $N$ ) of bound states.

The paper is organized as follows. In Sec. 2, the general principles are briefly described, which form the overall basis for the approach. Secs. 3 and 4 make an excursion to the theory of determinants, the benefits of which are described and illustrated in Sec. 5. Finally, Sec. 6 concludes the work.

## 2. RECONSTRUCTION OF REFLECTIONLESS POTENTIALS: UNIVERSAL RECIPE

Suppose we are given  $2N$  parameters for an unknown reflectionless potential  $V(x)$ : the positions of  $N$  discrete energy levels  $E_n = -C\kappa_n^2$  and  $N$  norming constants  $C_n$  ( $n = 1, 2, \dots, N$ ) for the Jost solution  $\Psi_1(i\kappa_n, x)$  of Eq. (1), so that  $\Psi_1(i\kappa_n, x) \rightarrow \exp(-\kappa_n x)$  as  $x \rightarrow +\infty$ , and

$$\int_{-\infty}^{\infty} \Psi_n^2(x) dx = 1, \quad \Psi_n \equiv C_n \Psi_1(i\kappa_n, x). \quad (5)$$

Then it can be shown [14] that

$$\Psi_n(x) = -\frac{1}{\Lambda_n(x)} \frac{\det[A^{(n)}]}{\det(A)}. \quad (6)$$

Here  $A$  is a symmetric matrix with the following elements:

$$A_{mn} = \delta_{mn} + \frac{\Lambda_m(x)\Lambda_n(x)}{\kappa_m + \kappa_n}, \quad (7)$$

$A^{(n)}$  is obtained from  $A$  by replacing the  $n$ th column with its derivative, and

$$\Lambda_n(x) \equiv C_n \exp(-\kappa_n x). \quad (8)$$

A simple formula can also be obtained for the potential [12, 14]:

$$V(x) = -2C \frac{d^2}{dx^2} \{ \ln [\det(A)] \}, \quad (9)$$

which is uniquely determined by parameters  $\kappa_n$  and  $C_n$ .

In principle, using Eqs. (7)-(9), one can reconstruct any reflectionless potential with given discrete energy levels and norming constants. In practice, however, the direct use of Eq. (9) is only justified if  $\det(A)$  can be easily calculated, which means that  $N$  must not be very large ( $N \leq 3$ ). With increasing  $N$  the problem becomes more and more troublesome, because the expansion of  $\det(A)$  contains  $N!$  terms. For example, if  $N = 10$ , there would be  $10! = 3628800$  terms, which means that ascertaining the potential is not at all trivial. Fortunately, as will be demonstrated in the forthcoming sections, there is no need to explicitly use Eq. (9). This general formula can be essentially simplified, so that the corresponding modified expansion will contain only  $2^{N-1}$  terms.

Note that the potential remains unchanged if  $\det(A)$  is multiplied by a function  $\exp(\alpha x + \beta)$ , where  $\alpha$  and  $\beta$  are arbitrary constants. Consequently, we can multiply, for example, any row of the initial matrix by a function  $\exp[\kappa_n(x - x_n)]$  ( $n = 1, 2, \dots, N$ ), where  $x_n$  are new parameters, equivalent to norming constants:

$$\exp(2\kappa_n x_n) \equiv \frac{C_n^2}{2\kappa_n}, \quad (10)$$

and thus, according to definition (8),

$$\frac{\Lambda_n^2(x)}{2\kappa_n} = \exp[-2\kappa_n(x - x_n)]. \quad (11)$$

As a result, we get another matrix which contains full information for reconstructing the potential according to Eq. (9):

$$\tilde{A}_N = \tilde{B}_N + \tilde{C}_N, \quad V(x) = -2C \frac{d^2}{dx^2} \{ \ln [\det(\tilde{A}_N)] \}. \quad (12)$$

Here

$$\tilde{B}_N \equiv \begin{pmatrix} e^{\kappa_1(x-x_1)} & 0 & \dots & 0 \\ 0 & e^{\kappa_2(x-x_2)} & \dots & 0 \\ 0 & \dots & \dots & \dots \\ 0 & \dots & \dots & e^{\kappa_N(x-x_N)} \end{pmatrix} \quad (13)$$

and

$$\tilde{C}_N \equiv \begin{pmatrix} e^{-\kappa_1(x-x_1)} & \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} e^{-\kappa_2(x-x_2)} & \dots & \frac{2\sqrt{\kappa_1\kappa_N}}{\kappa_1+\kappa_N} e^{-\kappa_N(x-x_N)} \\ \frac{2\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2} e^{-\kappa_1(x-x_1)} & e^{-\kappa_2(x-x_2)} & \dots & \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N} e^{-\kappa_N(x-x_N)} \\ \dots & \dots & \dots & \dots \\ \frac{2\sqrt{\kappa_1\kappa_N}}{\kappa_1+\kappa_N} e^{-\kappa_1(x-x_1)} & \frac{2\sqrt{\kappa_2\kappa_N}}{\kappa_2+\kappa_N} e^{-\kappa_2(x-x_2)} & \dots & e^{-\kappa_N(x-x_N)} \end{pmatrix}. \quad (14)$$

**Remark:** A subscript was added to denote the number of the bound states (and the rank of the matrix).

From now on, the determinants having the structure  $\det(A) \cdot \exp(\alpha x + \beta)$  with the elements  $A_{mm}$  defined in Eq. (7), will be sometimes called  $\tau$ -functions (as is common in soliton theory). In the next two sections it will be shown that such determinants can be easily calculated even for an arbitrarily large  $N$ .

### 3. GENERAL FORMULA FOR THE $\tau$ -FUNCTIONS

To evaluate a non-trivial determinant, one can use the Laplace expansion (see [15], p. 487) in terms of the fixed row (or column) indices. For example, choosing a set of indices  $m_1, m_2, \dots, m_k$  for an arbitrary  $N \times N$ -matrix  $A_N$ , so that  $1 \leq m_1 < m_2 < \dots \leq N$ , we get

$$\det(A_N) = \sum_{1 \leq n_1 \dots \leq N} \det A(m_1 m_2 \dots m_k | n_1 n_2 \dots n_k) \cdot \det \hat{A}(m_1 m_2 \dots m_k | n_1 n_2 \dots n_k). \quad (15)$$

Here  $A(m_1 m_2 \dots m_k | n_1 n_2 \dots n_k)$  is a  $k \times k$ -submatrix of  $A_N$  that lies on the intersection of rows  $m_1, m_2, \dots, m_k$  and columns  $n_1, n_2, \dots, n_k$ , while

$$\det \hat{A}(m_1 m_2 \dots m_k | n_1 n_2 \dots n_k) = (-1)^{m_1 + \dots + m_k + n_1 + \dots + n_k} M(m_1 m_2 \dots m_k | n_1 n_2 \dots n_k),$$

and  $M(m_1 m_2 \dots m_k | n_1 n_2 \dots n_k)$  is a minor obtained from  $\det(A_N)$  by deleting rows  $m_1, m_2, \dots, m_k$  and columns  $n_1, n_2, \dots, n_k$ .

Consequently, applying Eq. (15) to the  $\tau$ -function defined by Eqs. (12)-(14),

$$\tau_N \equiv \det(\tilde{A}_N) = a_0 \exp(\alpha_0) + \sum_{i=1}^N a_i \exp(\alpha_i) + \sum_{1 \leq i < j \leq N} a_{ij} \exp(\alpha_{ij}) + \sum_{1 \leq i < j < k \leq N} a_{ijk} \exp(\alpha_{ijk}) + \dots + a_{123 \dots N} \exp(\alpha_{123 \dots N}), \quad (16)$$

where the coefficients  $a_0, a_i, a_{ij}, a_{ijk}, \dots$  as well as the corresponding arguments of the exponents can be easily fixed with the help of Eqs. (12)-(14). Indeed,

$$\begin{aligned} \alpha_0 &= \sum_{l=1}^N \kappa_l(x-x_l), \quad \alpha_i = \sum_{l=1}^N (-1)^{\delta_{il}} \kappa_l(x-x_l), \quad \alpha_{ij} = \sum_{m=1}^N (-1)^{\delta_{im} + \delta_{jm}} \kappa_m(x-x_m), \\ \alpha_{ijk} &= \sum_{n=1}^N (-1)^{\delta_{in} + \delta_{jn} + \delta_{kn}} \kappa_n(x-x_n), \dots, \alpha_{123 \dots N} = \sum_{n=1}^N (-1)^{\delta_{1n} + \delta_{2n} + \dots + \delta_{Nn}} \kappa_n(x-x_n) = -\alpha_0, \end{aligned} \quad (17)$$

$\delta_{il}$  being the Kronecker's symbol.

To further simplify Eq. (16), let us group the terms into pairs, so that the arguments of the corresponding exponents differ only by sign. For example, the first pair is formed of the terms with coefficients  $a_0 = 1$  and

$$a_{123\dots N} \equiv A_0^2 = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} & \cdots & \frac{2\sqrt{\kappa_1 \kappa_N}}{\kappa_1 + \kappa_N} \\ \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} & 1 & \cdots & \frac{2\sqrt{\kappa_2 \kappa_N}}{\kappa_2 + \kappa_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2\sqrt{\kappa_1 \kappa_N}}{\kappa_1 + \kappa_N} & \frac{2\sqrt{\kappa_2 \kappa_N}}{\kappa_2 + \kappa_N} & \cdots & 1 \end{vmatrix}.$$

Note that

$$\det(\tilde{B}_N) = a_0 \exp(\alpha_0), \quad \det(\tilde{C}_N) = a_{123\dots N} \exp(-\alpha_0).$$

Here we defined a new coefficient  $A_0$ , whose subscript “0” emphasizes that the expression  $-\alpha_0 = -\kappa_1(x - x_1) - \kappa_2(x - x_2) - \dots - \kappa_N(x - x_N)$  contains no terms (0 terms) with plus sign. The same logic can be applied to all terms of Eq. (16). For example, the appropriate partner for the term  $a_1 \exp(\alpha_1)$  is  $a_{234\dots N} \exp(-\alpha_1)$ , where

$$a_1 = 1, \quad a_{234\dots N} \equiv A_1^2 = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_2 \kappa_3}}{\kappa_2 + \kappa_3} & \cdots & \frac{2\sqrt{\kappa_2 \kappa_N}}{\kappa_2 + \kappa_N} \\ \frac{2\sqrt{\kappa_2 \kappa_3}}{\kappa_2 + \kappa_3} & 1 & \cdots & \frac{2\sqrt{\kappa_3 \kappa_N}}{\kappa_3 + \kappa_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2\sqrt{\kappa_2 \kappa_N}}{\kappa_2 + \kappa_N} & \frac{2\sqrt{\kappa_3 \kappa_N}}{\kappa_3 + \kappa_N} & \cdots & 1 \end{vmatrix}.$$

Analogously, we can form a pair from  $a_{12} \exp(\alpha_{12})$  and  $a_{345\dots N} \exp(-\alpha_{12})$ , where

$$a_{12} = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} \\ \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} & 1 \end{vmatrix}, \quad a_{345\dots N} = \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_3 \kappa_4}}{\kappa_3 + \kappa_4} & \cdots & \frac{2\sqrt{\kappa_3 \kappa_N}}{\kappa_3 + \kappa_N} \\ \frac{2\sqrt{\kappa_3 \kappa_4}}{\kappa_3 + \kappa_4} & 1 & \cdots & \frac{2\sqrt{\kappa_4 \kappa_N}}{\kappa_4 + \kappa_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2\sqrt{\kappa_3 \kappa_N}}{\kappa_3 + \kappa_N} & \frac{2\sqrt{\kappa_4 \kappa_N}}{\kappa_4 + \kappa_N} & \cdots & 1 \end{vmatrix}.$$

The principle is simple: the terms  $a_{n_1 n_2 \dots n_i} \exp(\alpha_{n_1 n_2 \dots n_i})$  and  $a_{n_{i+1} n_{i+2} \dots n_N} \exp(-\alpha_{n_1 n_2 \dots n_i})$  are the partners in the sense explained above. In addition, as we will see below, it is convenient to define a relevant coefficient

$$A_{n_1 n_2 \dots n_i}^2 \equiv a_{n_1 n_2 \dots n_i} \cdot a_{n_{i+1} n_{i+2} \dots n_N}, \quad (18)$$

where the indices point at the terms with plus sign on the right side of the expression

$$-\alpha_{n_1 \dots n_i} = \kappa_{n_1}(x - x_{n_1}) + \dots + \kappa_{n_i}(x - x_{n_i}) - \kappa_{n_{i+1}}(x - x_{n_{i+1}}) - \dots \quad (19)$$

Looking at the structure of the matrix  $\tilde{A}_N$  (see Eqs. (12)-(14)) it is obvious that all these plus sign terms can only originate from the expansion of  $\det(\tilde{B}_N)$  and they correspond to the product  $(\tilde{B}_N)_{n_1 n_1} \cdot (\tilde{B}_N)_{n_2 n_2} \cdot \dots \cdot (\tilde{B}_N)_{n_i n_i}$ . The terms with minus sign on the right side of Eq. (19) are related to the expansion of  $\det(\tilde{C}_N)$ , without any contribution from  $\det(\tilde{B}_N)$ .

On the basis of the above arguments, the following conclusions can be made:

- All terms on the right side of Eq. (16) can be grouped into pairs. There is only one term,  $a_{123\dots N} \exp(-\alpha_0) = \det(\tilde{C}_N)$  (with partner  $\det(\tilde{B}_N) = a_0 \exp(\alpha_0)$ ) which is entirely formed of the elements of the matrix  $\tilde{C}_N$ . Any other term (both partners) contains some diagonal elements of the matrix  $\tilde{B}_N$  as well.
- Any term  $a_{n_1 n_2 \dots n_i} \exp(\alpha_{n_1 n_2 \dots n_i})$  ( $i = 1, 2, \dots$ ) can be obtained by replacing all elements of the rows and columns  $n_{i+1}, n_{i+2}, \dots, n_N$  of the matrix  $\tilde{C}_N$  with corresponding elements of the matrix  $\tilde{B}_N$  (mostly with zeros). As a result, one gets a modified matrix  $\tilde{C}_N$ , while  $a_{n_1 n_2 \dots n_i} \exp(\alpha_{n_1 n_2 \dots n_i}) = \det(\tilde{C}_N)$ . The Laplace expansion (15) of this determinant for the fixed rows  $n_{i+1}, n_{i+2}, \dots, n_N$  contains only one term!

- For any term  $a_{n_1 n_2 \dots n_i} \exp(\alpha_{n_1 n_2 \dots n_i})$  of the expansion (16) there is a partner

$$a_{n_{i+1} n_{i+2} \dots n_N} \exp(-\alpha_{n_1 n_2 \dots n_i}) = \det(\bar{C}_N),$$

where  $\bar{C}_N$  can be obtained by replacing all elements of the rows and columns  $n_1, n_2, \dots, n_i$  of the matrix  $\tilde{C}_N$  with corresponding elements of the matrix  $\tilde{B}_N$ . The Laplace expansion of  $\det(\bar{C}_N)$  for the rows  $n_1, n_2, \dots, n_i$  also contains only one term:

$$\det(\bar{C}_N) = \exp(-\alpha_{12\dots i}) \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_{n_{i+1}} \kappa_{n_{i+2}}}}{\kappa_{n_{i+1}} + \kappa_{n_{i+2}}} & \dots & \frac{2\sqrt{\kappa_{n_{i+1}} \kappa_{n_N}}}{\kappa_{n_{i+1}} + \kappa_{n_N}} \\ \frac{2\sqrt{\kappa_{n_{i+1}} \kappa_{n_{i+2}}}}{\kappa_{n_{i+1}} + \kappa_{n_{i+2}}} & 1 & \dots & \frac{2\sqrt{\kappa_{n_{i+2}} \kappa_{n_N}}}{\kappa_{n_{i+2}} + \kappa_{n_N}} \\ \dots & \dots & \dots & \dots \\ \frac{2\sqrt{\kappa_{n_{i+1}} \kappa_{n_N}}}{\kappa_{n_{i+1}} + \kappa_{n_N}} & \frac{2\sqrt{\kappa_{n_{i+2}} \kappa_{n_N}}}{\kappa_{n_{i+2}} + \kappa_{n_N}} & \dots & 1 \end{vmatrix}.$$

It means, for example, that

$$\begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} & \dots & \frac{2\sqrt{\kappa_1 \kappa_i}}{\kappa_1 + \kappa_i} \\ \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} & 1 & \dots & \frac{2\sqrt{\kappa_2 \kappa_i}}{\kappa_2 + \kappa_i} \\ \dots & \dots & \dots & \dots \\ \frac{2\sqrt{\kappa_1 \kappa_i}}{\kappa_1 + \kappa_i} & \frac{2\sqrt{\kappa_2 \kappa_i}}{\kappa_2 + \kappa_i} & \dots & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_{i+1} \kappa_{i+2}}}{\kappa_{i+1} + \kappa_{i+2}} & \dots & \frac{2\sqrt{\kappa_{i+1} \kappa_N}}{\kappa_{i+1} + \kappa_N} \\ \frac{2\sqrt{\kappa_{i+1} \kappa_{i+2}}}{\kappa_{i+1} + \kappa_{i+2}} & 1 & \dots & \frac{2\sqrt{\kappa_{i+2} \kappa_N}}{\kappa_{i+2} + \kappa_N} \\ \dots & \dots & \dots & \dots \\ \frac{2\sqrt{\kappa_{i+1} \kappa_N}}{\kappa_{i+1} + \kappa_N} & \frac{2\sqrt{\kappa_{i+2} \kappa_N}}{\kappa_{i+2} + \kappa_N} & \dots & 1 \end{vmatrix} = A_{12\dots i}^2 = a_{12\dots i} \cdot a_{i+1, i+2, \dots, N} = \quad (20)$$

- One should avoid re-use of the terms: an already existing pair must not be included again! It means that the members of the modified expansion (16) are identified by no more than  $[N/2]$  indices (square brackets denote integer part of  $N/2$ ). It is convenient to group the members on the basis of the number of indices, so that there will be  $0, 1, 2, \dots, [N/2]$  different indices.

As the final result of the above analysis, we get the following general formula:

$$\begin{aligned} \tau_N = \det(\tilde{A}_N) &= \sum_{i=0}^N A_i [\exp(\alpha_i + \beta_i) + \exp(-\alpha_i - \beta_i)] + \dots + \\ &= \sum_{1 \leq i_1 < \dots < i_{[N/2]} \leq N} A_{i_1 \dots i_{[N/2]}} [\exp(\alpha_{i_1 \dots i_{[N/2]}} + \beta_{i_1 \dots i_{[N/2]}}) + \exp(-\alpha_{i_1 \dots i_{[N/2]}} - \beta_{i_1 \dots i_{[N/2]}})] = \\ &= \sum_{i=0}^N 2A_i \cosh(\alpha_i + \beta_i) + \sum_{1 \leq i_1 < i_2 \leq N} 2A_{i_1 i_2} \cosh(\alpha_{i_1 i_2} + \beta_{i_1 i_2}) + \dots + \\ &= \sum_{1 \leq i_1 < \dots < i_{[N/2]} \leq N} 2A_{i_1 i_2 \dots i_{[N/2]}} \cosh(\alpha_{i_1 i_2 \dots i_{[N/2]}} + \beta_{i_1 i_2 \dots i_{[N/2]}}), \end{aligned} \quad (21)$$

where

$$\exp(\beta_{n_1 n_2 \dots n_i}) \equiv \frac{a_{n_1 n_2 \dots n_i}}{a_{n_{i+1} n_{i+2} \dots n_N}}. \quad (22)$$

It is easy to be convinced that the expansion (21) contains exactly  $2^{N-1}$  terms in total (apart from inessential factor 2). Indeed, there are  $N + 1 = C_0^N + C_1^N$  terms with just one index and  $C_k^N$  different terms with  $k > 1$  indices ( $i_1 i_2 \dots i_k$ ), while

$$C_0^N + C_1^N + C_2^N + C_3^N + \dots + C_{[N/2]}^N = \frac{(1+1)^N}{2} = 2^{N-1}, \quad C_k^N \equiv \frac{N!}{(N-k)!k!},$$

according to Newton's binomial theorem. Here we took into consideration that only half of this formal series is actually needed.

#### 4. ALTERNANTS OF $\tau$ -FUNCTIONS

We have shown that not only  $\det(\tilde{A}_N)$  itself but also the coefficients  $A_i, A_{i_1 i_2}, \dots, A_{i_1 i_2 \dots i_{\lfloor N/2 \rfloor}}$  in the expansion (21) are  $\tau$ -functions. Consequently, the solution of the inverse scattering problem has been reduced to evaluating a number of determinants

$$D(\kappa_1, \kappa_2, \dots, \kappa_n) \equiv \begin{vmatrix} 1 & \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} & \cdots & \frac{2\sqrt{\kappa_1 \kappa_n}}{\kappa_1 + \kappa_n} \\ \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} & 1 & \cdots & \frac{2\sqrt{\kappa_2 \kappa_n}}{\kappa_2 + \kappa_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2\sqrt{\kappa_1 \kappa_n}}{\kappa_1 + \kappa_n} & \frac{2\sqrt{\kappa_2 \kappa_n}}{\kappa_2 + \kappa_n} & \cdots & 1 \end{vmatrix}, \quad (23)$$

fixed by the parameters  $\kappa_1 < \kappa_2 < \dots < \kappa_n$ ,  $n$  being an appropriate natural number. We are now going to derive a simple formula for calculating such  $\tau$ -functions. First, let us set a one-to-one correspondence between each row of the determinant and a fixed parameter

$$q_i = \sqrt{\kappa_i} \quad (i = 1, 2, \dots, n). \quad (24)$$

For example, the modified elements of the first row of Eq. (23) will be

$$1 = \frac{2q_1\sqrt{\kappa_1}}{q_1^2 + \kappa_1}, \quad \frac{2\sqrt{\kappa_1 \kappa_2}}{\kappa_1 + \kappa_2} = \frac{2q_1\sqrt{\kappa_2}}{q_1^2 + \kappa_2}, \dots, \quad \frac{2\sqrt{\kappa_1 \kappa_n}}{\kappa_1 + \kappa_n} = \frac{2q_1\sqrt{\kappa_n}}{q_1^2 + \kappa_n}.$$

The usefulness of this trick soon becomes evident, although there seems to be only a formal change:

$$D(\kappa_1, \kappa_2, \dots, \kappa_n) \equiv \begin{vmatrix} \frac{2q_1\sqrt{\kappa_1}}{q_1^2 + \kappa_1} & \frac{2q_1\sqrt{\kappa_2}}{q_1^2 + \kappa_2} & \cdots & \frac{2q_1\sqrt{\kappa_n}}{q_1^2 + \kappa_n} \\ \frac{2q_2\sqrt{\kappa_1}}{q_2^2 + \kappa_1} & \frac{2q_2\sqrt{\kappa_2}}{q_2^2 + \kappa_2} & \cdots & \frac{2q_2\sqrt{\kappa_n}}{q_2^2 + \kappa_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2q_n\sqrt{\kappa_1}}{q_n^2 + \kappa_1} & \frac{2q_n\sqrt{\kappa_2}}{q_n^2 + \kappa_2} & \cdots & \frac{2q_n\sqrt{\kappa_n}}{q_n^2 + \kappa_n} \end{vmatrix} = 2^n \cdot \kappa_1 \kappa_2 \dots \kappa_n \begin{vmatrix} \frac{1}{q_1^2 + \kappa_1} & \frac{1}{q_1^2 + \kappa_2} & \cdots & \frac{1}{q_1^2 + \kappa_n} \\ \frac{1}{q_2^2 + \kappa_1} & \frac{1}{q_2^2 + \kappa_2} & \cdots & \frac{1}{q_2^2 + \kappa_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{q_n^2 + \kappa_1} & \frac{1}{q_n^2 + \kappa_2} & \cdots & \frac{1}{q_n^2 + \kappa_n} \end{vmatrix}. \quad (25)$$

Here we separated a factor  $2q_i$  from each row and  $\sqrt{\kappa_i}$  from each column, i.e.,  $2q_i\sqrt{\kappa_i} = 2\kappa_i$  ( $i = 1, 2, \dots, n$ ) from each such pair.

Next, let us transform Eq. (25) into a polynomial, multiplying each row by  $\prod_j (q_i^2 + \kappa_j)$  ( $i = 1, 2, \dots, n$ ).

The result is

$$D(\kappa_1, \kappa_2, \dots, \kappa_n) = \frac{1}{\prod_{1 \leq i < j \leq n} (\kappa_i + \kappa_j)^2} \times D_n, \quad (26)$$

where a new determinant

$$D_n \equiv \begin{vmatrix} \prod_{j \neq 1} (q_1^2 + \kappa_j) & \prod_{j \neq 2} (q_1^2 + \kappa_j) & \cdots & \prod_{j \neq n} (q_1^2 + \kappa_j) \\ \prod_{j \neq 1} (q_2^2 + \kappa_j) & \prod_{j \neq 2} (q_2^2 + \kappa_j) & \cdots & \prod_{j \neq n} (q_2^2 + \kappa_j) \\ \cdots & \cdots & \cdots & \cdots \\ \prod_{j \neq 1} (q_n^2 + \kappa_j) & \prod_{j \neq 2} (q_n^2 + \kappa_j) & \cdots & \prod_{j \neq n} (q_n^2 + \kappa_j) \end{vmatrix} \quad (27)$$

was introduced. As can be seen, the factor  $2^n \cdot \kappa_1 \kappa_2 \dots \kappa_n$  was canceled out from Eq. (26).

We can see that the elements of the columns of  $D_n$  correspond to different values of the same function, while any row is characterized by a single fixed parameter. Indeed, Eq. (27) can be expressed as

$$D_n = \begin{vmatrix} F_1(q_1) & F_2(q_1) & \dots & F_n(q_1) \\ F_1(q_2) & F_2(q_2) & \dots & F_n(q_2) \\ \dots & \dots & \dots & \dots \\ F_1(q_n) & F_2(q_n) & \dots & F_n(q_n) \end{vmatrix}, \quad (28)$$

where

$$\begin{aligned} F_1(x) &\equiv (x^2 + \kappa_2)(x^2 + \kappa_3) \dots (x^2 + \kappa_n), \\ F_2(x) &\equiv (x^2 + \kappa_1)(x^2 + \kappa_3) \dots (x^2 + \kappa_n), \\ F_3(x) &\equiv (x^2 + \kappa_1)(x^2 + \kappa_2)(x^2 + \kappa_4) \dots (x^2 + \kappa_n), \\ &\dots \\ F_n(x) &\equiv (x^2 + \kappa_1)(x^2 + \kappa_2)(x^2 + \kappa_3) \dots (x^2 + \kappa_{n-1}). \end{aligned} \quad (29)$$

A determinant that has a structure of Eq. (28) is called *alternant* (see [16], p. 161). The best known alternant is Vandermonde's determinant (for the same set of variables)

$$V_n \equiv \begin{vmatrix} 1 & q_1 & q_1^2 & \dots & q_1^{n-1} \\ 1 & q_2 & q_2^2 & \dots & q_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & q_{n-1} & q_{n-1}^2 & \dots & q_{n-1}^{n-1} \\ 1 & q_n & q_n^2 & \dots & q_n^{n-1} \end{vmatrix}, \quad (30)$$

which can be easily evaluated (see [17], p. 16):

$$V_n(q_1, q_2, \dots, q_n) = \prod_{1 \leq i < j \leq n} (q_j - q_i). \quad (31)$$

An important point is that the factor  $V_n(q_1, q_2, \dots, q_n)$  can be separated from any  $n$ th-order alternant. Indeed, the argument  $q_n$  may only appear in the  $n$ th row of Eq. (28): if we put it into any other row then the determinant would be identically zero. It means that  $D_n$  has a factor  $\prod_{i=1}^{n-1} (q_n - q_i)$ . Analogous reasoning applied to  $q_{n-1}$  shows that  $D_n$  also has a factor  $\prod_{i=1}^{n-2} (q_{n-1} - q_i)$ , etc. Putting it all together, we conclude that an  $n$ th-order alternant always has a factor  $V_n(q_1, q_2, \dots, q_n)$ .

To continue the analysis, let us recall some useful properties of the elementary symmetric functions:

$$\begin{aligned} \sigma_0 &\equiv 1, \\ \sigma_1 &\equiv q_1 + q_2 + \dots + q_n, \\ &\dots \\ \sigma_k &\equiv \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} q_{i_1} q_{i_2} \dots q_{i_k}, \\ &\dots \\ \sigma_n &\equiv q_1 q_2 \dots q_n. \end{aligned} \quad (32)$$

Here  $\sigma_k$  ( $k \geq 2$ ) is a sum of all possible products of exactly  $k$  variables arranged in ascending order of their indices. According to the Fundamental Theorem for symmetric polynomials (see [18], p. 312), any



such polynomial can be uniquely expressed as a polynomial in  $\sigma_1, \sigma_2, \dots, \sigma_n$ . This in turn is a basis for the following important theorem:

**Theorem 1:** Let  $|A_n|$  be an  $n$ th order alternant generated by the functions

$$F_j(x) = a_{0j} + a_{1j} \cdot x + a_{2j} \cdot x^2 + \dots + a_{nj} \cdot x^n, \quad j = 1, 2, \dots, n, \quad (33)$$

where the parameters  $a_{ij}$  do not depend on  $x$ , and define

$$S_k \equiv (-1)^k \sigma_k(q_1, q_2, \dots, q_n). \quad (34)$$

Then

$$\frac{|A_n|}{V_n(q_1, q_2, \dots, q_n)} = \begin{vmatrix} a_{01} & a_{11} & a_{21} & \dots & a_{n1} \\ a_{02} & a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{0n} & a_{1n} & a_{2n} & \dots & a_{nn} \\ S_n & S_{n-1} & S_{n-2} & \dots & S_0 \end{vmatrix} \equiv \Delta_{n+1}. \quad (35)$$

**Proof:** Let us introduce an auxiliary polynomial

$$F(x) \equiv (x - q_1)(x - q_2) \dots (x - q_n) \quad (36)$$

and form an  $(n + 1)$ th order Vandermonde's determinant, adding a new (arbitrary) variable  $q_{n+1}$ , so that

$$V_{n+1}(q_1, q_2, \dots, q_n, q_{n+1}) = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ q_1 & q_2 & \dots & q_n & q_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ q_1^{n-1} & q_2^{n-1} & \dots & q_n^{n-1} & q_{n+1}^{n-1} \\ q_1^n & q_2^n & \dots & q_n^n & q_{n+1}^n \end{vmatrix}$$

(compared with Eq. (30), the row and column indices are interchanged). Multiplying  $\Delta_{n+1}$  by  $V_{n+1}$  and using Eqs. (33), (34), (36), we get

$$\Delta_{n+1} \times V_{n+1} = \begin{vmatrix} F_1(q_1) & F_1(q_2) & \dots & F_1(q_n) & F_1(q_{n+1}) \\ F_2(q_1) & F_2(q_2) & \dots & F_2(q_n) & F_2(q_{n+1}) \\ \dots & \dots & \dots & \dots & \dots \\ F_n(q_1) & F_n(q_2) & \dots & F_n(q_n) & F_n(q_{n+1}) \\ F(q_1) & F(q_2) & \dots & F(q_n) & F(q_{n+1}) \end{vmatrix} = \begin{vmatrix} F_1(q_1) & F_1(q_2) & \dots & F_1(q_n) & F_1(q_{n+1}) \\ F_2(q_1) & F_2(q_2) & \dots & F_2(q_n) & F_2(q_{n+1}) \\ \dots & \dots & \dots & \dots & \dots \\ F_n(q_1) & F_n(q_2) & \dots & F_n(q_n) & F_n(q_{n+1}) \\ 0 & 0 & \dots & 0 & F(q_{n+1}) \end{vmatrix}$$

or, in a more compact form,

$$\Delta_{n+1} \times V_{n+1} = (q_{n+1} - q_1) \dots (q_{n+1} - q_n) \times |A_n|, \quad (37)$$

where we took into consideration that  $F(q_i) = 0$  if  $i = 1, 2, \dots, n$ .

From Eq. (31) one concludes that

$$V_{n+1} = (q_{n+1} - q_1)(q_{n+1} - q_2) \dots (q_{n+1} - q_n) \cdot V_n,$$

so that both sides of Eq. (37) contain a common divisor  $(q_{n+1} - q_1)(q_{n+1} - q_2) \dots (q_{n+1} - q_n) \neq 0$ . Consequently,  $\Delta_{n+1} V_n = |A_n|$ , q. e. d.

#### 4.1. Relationship to the inverse scattering problem

Let us apply Theorem 1 to the alternant (28) generated by the functions (29), which are polynomials in a variable  $x^2$ . Using Eqs. (33) and (34), one gets

$$\begin{aligned}
 a_{01} &= \sigma_{n-1}(\kappa_2, \kappa_3, \dots, \kappa_n), \\
 a_{11} &= \sigma_{n-2}(\kappa_2, \kappa_3, \dots, \kappa_n), \dots, a_{n-1,1} = 1, a_{n1} = 0, \\
 a_{02} &= \sigma_{n-1}(\kappa_1, \kappa_3, \dots, \kappa_n), \\
 a_{12} &= \sigma_{n-2}(\kappa_1, \kappa_3, \dots, \kappa_n), \dots, a_{n-1,2} = 1, a_{n2} = 0, \\
 &\dots \\
 a_{0n} &= \sigma_{n-1}(\kappa_1, \kappa_2, \dots, \kappa_{n-1}), \\
 a_{1n} &= \sigma_{n-2}(\kappa_1, \kappa_2, \dots, \kappa_{n-1}), \dots, a_{n-1,n} = 1, a_{nn} = 0.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 D_n &= V_n(q_1, q_2, \dots, q_n) \times |R_n|, \tag{38} \\
 |R_n| &\equiv \begin{vmatrix} \sigma_{n-1}(\kappa_2, \kappa_3, \dots, \kappa_n) & \sigma_{n-2}(\kappa_2, \kappa_3, \dots, \kappa_n) & \dots & 1 & 0 \\ \sigma_{n-1}(\kappa_1, \kappa_3, \dots, \kappa_n) & \sigma_{n-2}(\kappa_1, \kappa_3, \dots, \kappa_n) & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{n-1}(\kappa_1, \kappa_2, \dots, \kappa_{n-1}) & \sigma_{n-2}(\kappa_1, \kappa_2, \dots, \kappa_{n-1}) & \dots & 1 & 0 \\ S_n & S_{n-1} & \dots & S_1 & S_0 \end{vmatrix} = \\
 &\begin{vmatrix} \sigma_{n-1}(\kappa_2, \kappa_3, \dots, \kappa_n) & \sigma_{n-2}(\kappa_2, \kappa_3, \dots, \kappa_n) & \dots & 1 \\ \sigma_{n-1}(\kappa_1, \kappa_3, \dots, \kappa_n) & \sigma_{n-2}(\kappa_1, \kappa_3, \dots, \kappa_n) & \dots & 1 \\ \dots & \dots & \dots & \dots \\ \sigma_{n-1}(\kappa_1, \kappa_2, \dots, \kappa_{n-1}) & \sigma_{n-2}(\kappa_1, \kappa_2, \dots, \kappa_{n-1}) & \dots & 1 \end{vmatrix}.
 \end{aligned}$$

Here we defined a new determinant  $|R_n|$  which seems to be another alternant, so we can apply Theorem 1 to evaluate it. To be convinced that  $|R_n|$  is indeed an alternant, we have to specify the generating functions. Obviously,

$$F_n(x) = 1 = \sigma_0(\kappa_1, \kappa_2, \dots, \kappa_n), F_{n-1}(x) = \sigma_1(\kappa_1, \kappa_2, \dots, \kappa_n) - x. \tag{39}$$

Also, it is easy to prove that

$$F_{n-2}(x) = \sigma_2 - xF_{n-1}(x) = \sigma_2 - \sigma_1 x + x^2, \tag{40}$$

where we dropped the arguments to get a more compact formula. Let us agree that if no arguments are explicitly given then the corresponding function depends on  $n$  arguments:  $\kappa_1, \kappa_2, \dots, \kappa_n$ . Taking, e.g.,  $x = \kappa_1$  we can check the validity of Eq. (40). Indeed,

$$\begin{aligned}
 F_{n-2}(\kappa_1) &= \sigma_2 - \sigma_1 \kappa_1 + \kappa_1^2 = \\
 &\sum_{1 \leq i_1 < i_2 \leq n} (\kappa_{i_1} \kappa_{i_2}) - \kappa_1 (\kappa_1 + \kappa_2 + \dots + \kappa_n) + \kappa_1^2 = \sigma_{n-1}(\kappa_2, \kappa_3, \dots, \kappa_n),
 \end{aligned}$$

as needed according to Eq. (38). Continuing in the same manner, we get the following result:

$$\begin{aligned}
 F_{n-3}(x) &= \sigma_3 - xF_{n-2}(x) = \sigma_3 - \sigma_2 x + \sigma_1 x^2 - x^3, \\
 &\dots \\
 F_{n-i}(x) &= \sigma_{n-i} - xF_{n-i-1}(x) = \sum_{j=0}^i (-1)^j \sigma_{i-j} x^j \rightarrow \\
 F_k(x) &= \sum_{j=0}^{n-k} (-1)^j \sigma_{n-k-j} x^j = \sum_{j=0}^{n-k} a_{jk} x^j, \tag{41}
 \end{aligned}$$

$$a_{jk} = \begin{cases} (-1)^j \sigma_{n-k-j}, & j \leq n-k, \\ 0, & j > n-k. \end{cases} \tag{42}$$

Thus  $|R_n|$  is indeed an alternant with generating functions (33). Consequently, according to Eqs. (34), (35) and (42),

$$\frac{|R_n|}{V_n} = \begin{vmatrix} a_{01} & a_{11} & a_{21} & \dots & a_{n1} \\ a_{02} & a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{0n} & a_{1n} & a_{2n} & \dots & a_{nn} \\ S_n & S_{n-1} & S_{n-2} & \dots & S_0 \end{vmatrix} =$$

$$\begin{vmatrix} \sigma_{n-1} & -\sigma_{n-2} & \sigma_{n-3} & \dots & (-1)^{n-2}\sigma_1 & (-1)^{n-1} & 0 \\ \sigma_{n-2} & -\sigma_{n-3} & \sigma_{n-4} & \dots & (-1)^{n-2} & 0 & 0 \\ \sigma_{n-3} & -\sigma_{n-4} & \sigma_{n-5} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 & 0 \\ \sigma_1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ (-1)^n \sigma_n & (-1)^{n-1} \sigma_{n-1} & (-1)^{n-1} \sigma_{n-2} & \dots & \sigma_2 & -\sigma_1 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} \sigma_{n-1} & -\sigma_{n-2} & \sigma_{n-3} & \dots & (-1)^{n-2} \sigma_1 & (-1)^{n-1} \\ \sigma_{n-2} & -\sigma_{n-3} & \sigma_{n-4} & \dots & (-1)^{n-2} & 0 \\ \sigma_{n-3} & -\sigma_{n-4} & \sigma_{n-5} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & 0 & 0 \\ \sigma_1 & -1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{vmatrix}, \quad (43)$$

which means that

$$\frac{|R_n|}{V_n} = 1. \quad (44)$$

Indeed, we can transform Eq. (43), repeatedly using cofactor expansion in terms of the last column and applying the general definition

$$\det(A) = \sum_p \sigma(p) a_{1p_1} a_{2p_2} \dots a_{np_n}. \quad (45)$$

Here the sum involves all possible permutations  $p = (p_1, p_2, \dots, p_n)$  of indices  $(1, 2, \dots, n)$ ,  $\sigma(p) = (-1)^{N_p}$  and  $N_p$  is the number of pairwise interchanges needed to restore the natural order  $(1, 2, \dots, n)$ . For example,  $\sigma(3, 2, 1) = -1$ , but  $\sigma(4, 3, 2, 1) = 1$ . As a result of the described operation, we obtain

$$\frac{|R_n|}{V_n} = \sigma(n, n-1, \dots, 1) \cdot (-1)^{\lfloor n/2 \rfloor}.$$

On the other hand,  $\sigma(n, n-1, \dots, 1) = (-1)^{\lfloor n/2 \rfloor}$ , since  $N_p = \lfloor n/2 \rfloor$ . Consequently,

$$|R_n| = V_n \cdot (-1)^{\lfloor n/2 \rfloor} \cdot (-1)^{\lfloor n/2 \rfloor} = V_n.$$

In summary, we have obtained a very simple and universal recipe for calculating determinants defined by Eq. (27):

$$D_n = (V_n)^2 = \prod_{1 \leq i < j \leq n} (\kappa_j - \kappa_i)^2. \quad (46)$$

Combining Eqs. (26) and (46), we can formulate a general and important result:

**Theorem 2:** Let  $\kappa_1, \kappa_2, \dots, \kappa_n$  be arbitrary positive real numbers arranged in the ascending order, so that  $\kappa_1 < \kappa_2 < \dots < \kappa_n$ , and let  $D(\kappa_1, \kappa_2, \dots, \kappa_n)$  be a determinant, defined by Eq. (23). Then

$$D(\kappa_1, \kappa_2, \dots, \kappa_n) = \prod_{1 \leq i < j \leq n} \left( \frac{\kappa_j - \kappa_i}{\kappa_j + \kappa_i} \right)^2, \quad (47)$$

where the product contains all possible combinations of the pairs  $(\kappa_j, \kappa_i)$  satisfying the condition  $1 \leq i < j \leq n$ .

## 5. SYMMETRIC REFLECTIONLESS POTENTIALS

The excursion to theory of determinants concluded with a surprisingly simple final result. Indeed, as is seen from Eqs. (18), (20) and (22), all coefficients in Eq. (21) can be evaluated with the help of Eq. (47), which means that the general formula for  $\tau$ -functions can be essentially simplified. For example,

$$A_0 = \sqrt{D(\kappa_1, \kappa_2, \dots, \kappa_n)} = \prod_{1 \leq i < j \leq n} \left( \frac{\kappa_j - \kappa_i}{\kappa_j + \kappa_i} \right), \quad (48)$$

$$\frac{A_i}{A_0} = \sqrt{\frac{D(\kappa_1, \kappa_2, \dots, \kappa_{i-1}, \kappa_{i+1}, \kappa_{i+2}, \dots, \kappa_n)}{D(\kappa_1, \kappa_2, \dots, \kappa_n)}} = \prod_{j \neq i} \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right|, \quad (49)$$

etc. Thus Eq. (21) transforms to

$$\begin{aligned} \tau_N = \frac{\det(\tilde{A}_N)}{2A_0} &= \cosh(\alpha_0 + \beta_0) + \sum_i \prod_{j \neq i} \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right| \cosh(\alpha_i + \beta_i) + \\ &\sum_{1 \leq i_1 < i_2 \leq N} \prod_{j \neq i_1, i_2} \left| \frac{\kappa_j + \kappa_{i_1}}{\kappa_j - \kappa_{i_1}} \right| \cdot \left| \frac{\kappa_j + \kappa_{i_2}}{\kappa_j - \kappa_{i_2}} \right| \cosh(\alpha_{i_1 i_2} + \beta_{i_1 i_2}) + \dots + \\ &\sum_{1 \leq i_1, \dots, i_{\lfloor N/2 \rfloor} \leq N} \prod_{j \neq i_1, i_2, \dots, i_{\lfloor N/2 \rfloor}} \left| \frac{\kappa_j + \kappa_{i_1}}{\kappa_j - \kappa_{i_1}} \right| \cdot \left| \frac{\kappa_j + \kappa_{i_2}}{\kappa_j - \kappa_{i_2}} \right| \dots \left| \frac{\kappa_j + \kappa_{i_{\lfloor N/2 \rfloor}}}{\kappa_j - \kappa_{i_{\lfloor N/2 \rfloor}}} \right| \cosh(\alpha_{i_1 \dots i_{\lfloor N/2 \rfloor}} + \beta_{i_1 \dots i_{\lfloor N/2 \rfloor}}), \end{aligned} \quad (50)$$

and Eq. (12) can be rewritten as

$$V(x) = -2C \frac{d^2}{dx^2} \{ \ln \tau_N(x) \}. \quad (51)$$

Eqs. (50)-(51) express the main result of this paper.

In general, as mentioned, the reflectionless potential is uniquely determined if  $2N$  parameters  $\kappa_n$  and  $C_n$  ( $n = 1, 2, \dots, N$ ) are known. However, if one sets an additional constraint

$$V(-x) = V(x)$$

then the number of necessary input parameters is twofold reduced. In other words, a symmetric reflectionless potential is uniquely determined by its  $N$  binding energies [19]  $E_n = -C\kappa_n^2$ . Let us analyze this in more detail.

Obviously, Eq. (50) can only be symmetric if the arguments of all cosh functions are of the linear form  $ax + b$  with  $b \equiv 0$ . It means, for example, that

$$\sum_{i=1}^N \kappa_i x_i = \beta_0 = \sum_{1 \leq i < j \leq N} c_{ij}, \quad (52)$$

$$\kappa_1 x_1 - \kappa_2 x_2 - \dots - \kappa_N x_N = 2\kappa_1 x_1 - \beta_0 = -\beta_1 = \sum_{j=2}^N c_{1j} - \beta_0, \quad (53)$$

where

$$c_{ij} \equiv \ln \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right|, \quad (54)$$

and consequently,

$$2\kappa_1 x_1 = \sum_{j=2}^N c_{1j}.$$

Using Eqs. (17) and Eq. (22), we get a similar expression for any other combination  $\kappa_i x_i$ . From Eqs. (10) and (54) we therefore obtain the following symmetricity conditions:

$$\exp(2\kappa_i x_i) = \frac{C_i^2}{2\kappa_i} = \prod_{j \neq i}^N \left| \frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i} \right|, \quad i = 1, 2, \dots, N \geq 2. \quad (55)$$

If  $N = 1$  then

$$\exp(2\kappa_1 x_1) = \frac{C_1^2}{2\kappa_1} = 1. \quad (56)$$

It can be easily shown that Eqs. (55) are indeed the symmetricity conditions for  $\tau_N$  and  $V(x)$ . To this end, in full analogy with Eq. (53), we can write

$$\begin{aligned} \kappa_1 x_1 - \kappa_2 x_2 - \dots - \kappa_N x_N &= \sum_{j \neq 1}^N c_{1j} - \beta_0, \\ -\kappa_1 x_1 + \kappa_2 x_2 - \dots - \kappa_N x_N &= \sum_{j \neq 2}^N c_{2j} - \beta_0, \\ &\dots \\ -\kappa_1 x_1 + \kappa_2 x_2 - \dots + \kappa_N x_N &= \sum_{j \neq N}^N c_{Nj} - \beta_0. \end{aligned} \quad (57)$$

Summing these equations, we get

$$-(N-2) \sum_{i=1}^N \kappa_i x_i = -(N-2)\beta_0,$$

which coincides with Eq. (52).

**Remark:** If  $N = 2$  then the first two equations of the system (57) are not linearly independent, since

$$\beta_1 = \beta_0 - \sum_{j \neq 1}^N c_{1j} = \beta_0 - \sum_{j \neq 2}^N c_{2j} = \beta_2 = 0,$$

so that  $\kappa_1 x_1 = \kappa_2 x_2$ . Consequently, in this (and only in this) special case Eqs. (52)-(53) must be treated as the actual symmetricity conditions, while Eq. (55) still remains valid.

The next step is to complement Eq. (57), for example, with another condition

$$2(\kappa_1 x_1 + \kappa_2 x_2) = \sum_{j \neq 1}^N c_{1j} + \sum_{j \neq 2}^N c_{2j}, \quad (58)$$

which is a direct conclusion from Eq. (55). Thus

$$\begin{aligned} 2(\kappa_1 x_1 + \kappa_2 x_2) - \sum_{i=1}^N \kappa_i x_i &= \kappa_1 x_1 + \kappa_2 x_2 - \kappa_3 x_3 - \dots - \kappa_N x_N = \\ &= \sum_{j \neq 1}^N c_{1j} + \sum_{j \neq 2}^N c_{2j} - \beta_0 = \beta_{12}, \end{aligned}$$

which means that

$$\cosh(\alpha_{12} + \beta_{12}) = \cosh[(\kappa_1 + \kappa_2 - \kappa_3 - \dots - \kappa_N) x]$$

is a symmetric function. Analogously, one can prove that any other term  $\cosh(\alpha_{i_1 i_2 \dots} + \beta_{i_1 i_2 \dots})$  in Eq. (50) is a symmetric function as well. This in turn proves that the norming constants  $C_n$  of a symmetric reflectionless potential are uniquely determined by the given binding energies  $E_n$ .

### 5.1. Few practical examples

To illustrate the results, let us take, for example,  $N = 4$ . Then Eq. (50) reads

$$\begin{aligned}
T_4 = & \cosh(\alpha_0 + \beta_0) + \left(\frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1}\right) \left(\frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1}\right) \left(\frac{\kappa_4 + \kappa_1}{\kappa_4 - \kappa_1}\right) \cosh(\alpha_1 + \beta_1) + \\
& \left(\frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1}\right) \left(\frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2}\right) \left(\frac{\kappa_4 + \kappa_2}{\kappa_4 - \kappa_2}\right) \cosh(\alpha_2 + \beta_2) + \left(\frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1}\right) \left(\frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2}\right) \times \\
& \left(\frac{\kappa_4 + \kappa_3}{\kappa_4 - \kappa_3}\right) \cosh(\alpha_3 + \beta_3) + \left(\frac{\kappa_4 + \kappa_1}{\kappa_4 - \kappa_1}\right) \left(\frac{\kappa_4 + \kappa_2}{\kappa_4 - \kappa_2}\right) \left(\frac{\kappa_4 + \kappa_3}{\kappa_4 - \kappa_3}\right) \cosh(\alpha_4 + \beta_4) + \\
& \left(\frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1}\right) \left(\frac{\kappa_4 + \kappa_1}{\kappa_4 - \kappa_1}\right) \left(\frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2}\right) \left(\frac{\kappa_4 + \kappa_2}{\kappa_4 - \kappa_2}\right) \cosh(\alpha_{12} + \beta_{12}) + \\
& \left(\frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1}\right) \left(\frac{\kappa_4 + \kappa_1}{\kappa_4 - \kappa_1}\right) \left(\frac{\kappa_3 + \kappa_2}{\kappa_3 - \kappa_2}\right) \left(\frac{\kappa_4 + \kappa_3}{\kappa_4 - \kappa_3}\right) \cosh(\alpha_{13} + \beta_{13}) + \\
& \left(\frac{\kappa_2 + \kappa_1}{\kappa_2 - \kappa_1}\right) \left(\frac{\kappa_3 + \kappa_1}{\kappa_3 - \kappa_1}\right) \left(\frac{\kappa_4 + \kappa_2}{\kappa_4 - \kappa_2}\right) \left(\frac{\kappa_4 + \kappa_3}{\kappa_4 - \kappa_3}\right) \cosh(\alpha_{14} + \beta_{14}),
\end{aligned} \tag{59}$$

while the symmetricity conditions, according to Eqs. (53)-(55), can be given as

$$\begin{aligned}
\kappa_1 x_1 - \kappa_2 x_2 - \kappa_3 x_3 - \kappa_4 x_4 = -\beta_1 &= \ln \left( \frac{\kappa_3 - \kappa_2}{\kappa_3 + \kappa_2} \frac{\kappa_4 - \kappa_2}{\kappa_4 + \kappa_2} \frac{\kappa_4 - \kappa_3}{\kappa_4 + \kappa_3} \right), \\
-\kappa_1 x_1 + \kappa_2 x_2 - \kappa_3 x_3 - \kappa_4 x_4 = -\beta_2 &= \ln \left( \frac{\kappa_3 - \kappa_1}{\kappa_3 + \kappa_1} \frac{\kappa_4 - \kappa_1}{\kappa_4 + \kappa_1} \frac{\kappa_4 - \kappa_3}{\kappa_4 + \kappa_3} \right), \\
-\kappa_1 x_1 - \kappa_2 x_2 + \kappa_3 x_3 - \kappa_4 x_4 = -\beta_3 &= \ln \left( \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1} \frac{\kappa_4 - \kappa_1}{\kappa_4 + \kappa_1} \frac{\kappa_4 - \kappa_2}{\kappa_4 + \kappa_2} \right), \\
-\kappa_1 x_1 - \kappa_2 x_2 - \kappa_3 x_3 + \kappa_4 x_4 = -\beta_4 &= \ln \left( \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1} \frac{\kappa_3 - \kappa_1}{\kappa_3 + \kappa_1} \frac{\kappa_3 - \kappa_2}{\kappa_3 + \kappa_2} \right).
\end{aligned} \tag{60}$$

Summing the corresponding sides of Eq. (60) we get

$$\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3 + \kappa_4 x_4 + \beta_0 = 0,$$

which means that

$$\cosh(\alpha_0 + \beta_0) = \cosh[(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)x]$$

is a symmetric function.

Analogously, subtracting the sides of the last two equations from the corresponding sides of the first two equations of (60), we get

$$\kappa_1 x_1 + \kappa_2 x_2 - \kappa_3 x_3 - \kappa_4 x_4 + \beta_{12} = 0,$$

which means that  $\cosh(\alpha_{12} + \beta_{12})$  is a symmetric function. Continuing in a similar manner, it is easy to be convinced that  $\cosh(\alpha_{13} + \beta_{13})$  and  $\cosh(\alpha_{14} + \beta_{14})$  are symmetric functions as well.

**Example 1:** To be more specific, let  $x_0 = 1/\kappa_1$  be the length unit and  $E_0 = C\kappa_1^2$ , the energy unit (i.e.,  $x_0 = 1$  and  $E_0 = 1$ ). The simplest and the best known symmetric potential then corresponds to

$$\kappa_n = n, \quad n = 1, 2, 3, 4.$$

Therefore, according to Eq. (59),

$$\begin{aligned}
T_4 = & \cosh[(\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4)x] + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} \cosh[(\kappa_1 - \kappa_2 - \kappa_3 - \kappa_4)x] + \frac{3 \cdot 5 \cdot 6}{1 \cdot 1 \cdot 2} \times \\
& \cosh[(\kappa_2 - \kappa_1 - \kappa_3 - \kappa_4)x] + \frac{4 \cdot 5 \cdot 7}{2 \cdot 1 \cdot 1} \cosh[(\kappa_3 - \kappa_1 - \kappa_2 - \kappa_4)x] + \frac{5 \cdot 6 \cdot 7}{3 \cdot 2 \cdot 1} \times \\
& \cosh[(\kappa_4 - \kappa_1 - \kappa_2 - \kappa_3)x] + \frac{4 \cdot 5 \cdot 5 \cdot 6}{2 \cdot 3 \cdot 1 \cdot 2} \cosh[(\kappa_1 + \kappa_2 - \kappa_3 - \kappa_4)x] + \frac{3 \cdot 5 \cdot 5 \cdot 7}{1 \cdot 3 \cdot 1 \cdot 1} \times \\
& \cosh[(\kappa_1 + \kappa_3 - \kappa_2 - \kappa_4)x] + \frac{3 \cdot 4 \cdot 6 \cdot 7}{1 \cdot 2 \cdot 2 \cdot 1} \cosh[(\kappa_1 + \kappa_4 - \kappa_2 - \kappa_3)x] = \\
& \cosh(10x) + 10 \cosh(8x) + 45 \cosh(6x) + 120 \cosh(4x) + 210 \cosh(2x) + 126.
\end{aligned} \tag{61}$$

At first sight Eq. (61) may seem impractical. However, using the standard transformation formulas (obtained from corresponding trigonometric formulas by replacing  $x \rightarrow ix$ ):

$$\begin{aligned}
\cosh(10x) &= 512 \cosh^{10}(x) - 1280 \cosh^8(x) + 1120 \cosh^6(x) - 400 \cosh^4(x) + 50 \cosh^2(x) - 1, \\
\cosh(8x) &= 128 \cosh^8(x) - 256 \cosh^6(x) + 160 \cosh^4(x) - 32 \cosh^2(x) + 1, \\
\cosh(6x) &= 32 \cosh^6(x) - 48 \cosh^4(x) + 18 \cosh^2(x) - 1, \\
\cosh(4x) &= 8 \cosh^4(x) - 8 \cosh^2(x) + 1, \quad \cosh(2x) = 2 \cosh^2(x) - 1,
\end{aligned}$$

the result is as follows:

$$T_4 = 512 \cosh^{10}(x).$$

Thus

$$[\ln T_4(x)]' = \frac{T_4'}{T_4} = 10 \tanh(x)$$

and

$$V(x) = -\frac{20}{\cosh^2(x)} = -\frac{N(N+1)}{\cosh^2(x)},$$

exactly as needed.

**Example 2:** Next, let us construct a reflectionless approximant to a symmetric rectangular potential with four energy levels (see Fig. 1). These levels can be determined from (see, e.g., [20], Sec. II.9)

$$\kappa_i x_i \tan(\kappa_i x_i) = \sqrt{\frac{U_0 a^2}{C} - (\kappa_i x_i)^2}, \tag{62}$$

$$-\frac{\kappa_i x_i}{\tan(\kappa_i x_i)} = \sqrt{\frac{U_0 a^2}{C} - (\kappa_i x_i)^2}, \tag{63}$$

where  $a$  and  $U_0$  denote the half-width and the depth of the potential well, respectively. Eq. (62) fixes the symmetric and (63) – the antisymmetric solutions to the Schrödinger equation. Again, it is convenient to use dimensionless units for the length and energy, taking  $x_0 = a = 1$  and  $E_0 = C/a^2 = 1$ . In addition, let us fix

$$\sqrt{\frac{U_0 a^2}{C}} = 5.$$

Then the system has four discrete levels (as assumed) corresponding to

$$\begin{aligned}
\kappa_1 &= 1.3064400089, \quad \kappa_2 = 2.5957390789, \\
\kappa_3 &= 3.8374671080, \quad \kappa_4 = 4.9062951521.
\end{aligned} \tag{64}$$

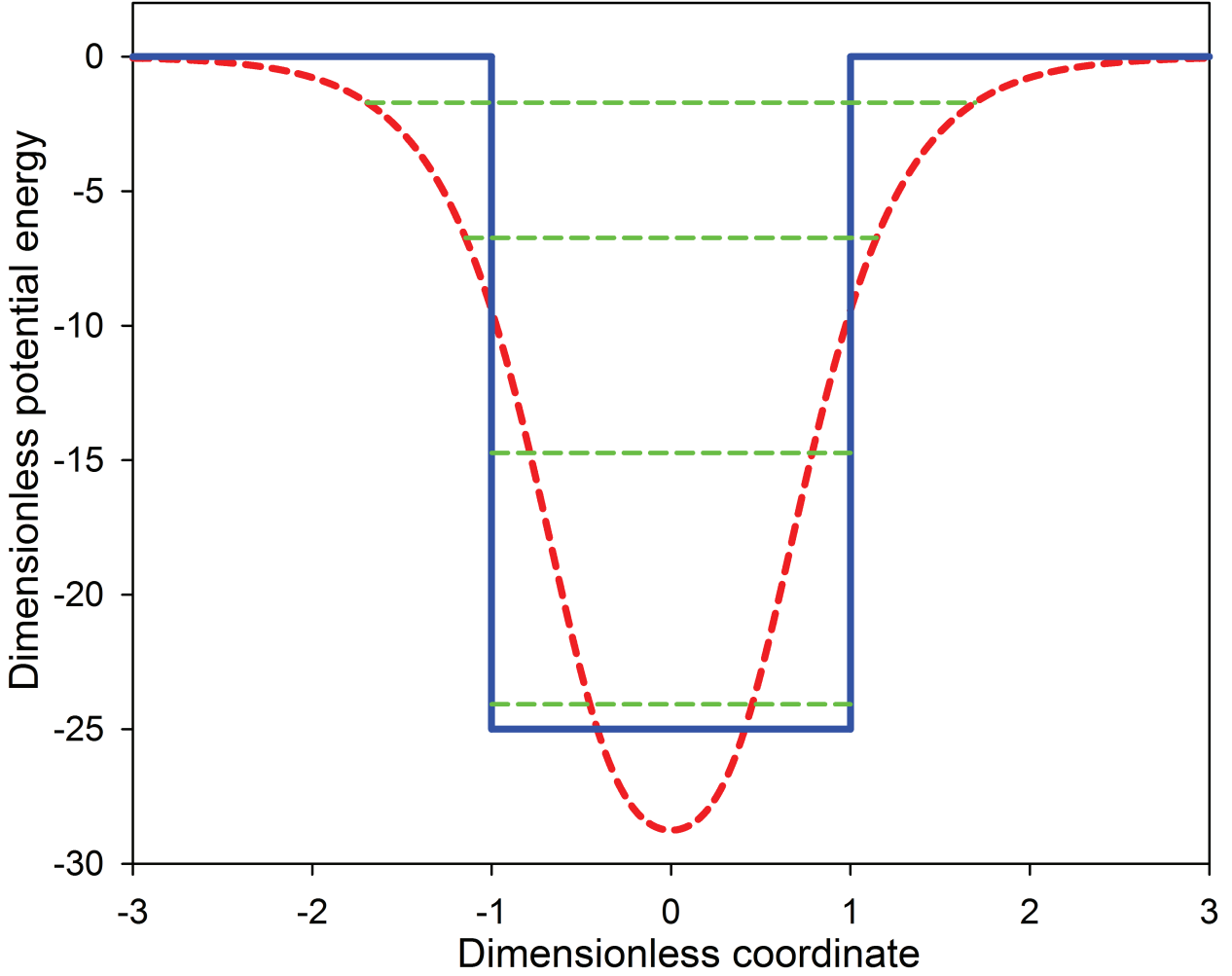


Fig. 1. The isospectral reflectionless approximant (red dashed line) to a symmetric rectangular potential (blue solid line). The common energy levels are shown by green dashed lines.

In this case Eq. (59) cannot be further simplified, but this is not any serious problem. Indeed, let us define the coefficients  $A_i$  and  $B_i$ , such that

$$T_4 = \sum_i A_i \cosh(B_i x).$$

Then the corresponding potential becomes

$$V(x) = -2C \left\{ \frac{\sum_i A_i B_i^2 \cosh(B_i x)}{\sum_i A_i \cosh(B_i x)} - \frac{\left[ \frac{\sum_i A_i B_i \sinh(B_i x)}{\sum_i A_i \cosh(B_i x)} \right]^2}{\sum_i A_i \cosh(B_i x)} \right\}. \quad (65)$$

The result for the input data (64) can be seen in Fig. 1.

**Example 3:** Fig. 2 demonstrates two isospectral potentials corresponding to the following set of input parameters:

$$\kappa_1 = 1/8, \kappa_2 = 3/8, \kappa_3 = 5/8, \kappa_4 = 7/8, \quad (66)$$

the four energy levels being  $E_n = -C\kappa_n^2$  ( $n = 1, 2, 3, 4$ ) as previously. The solid curve in this figure corre-



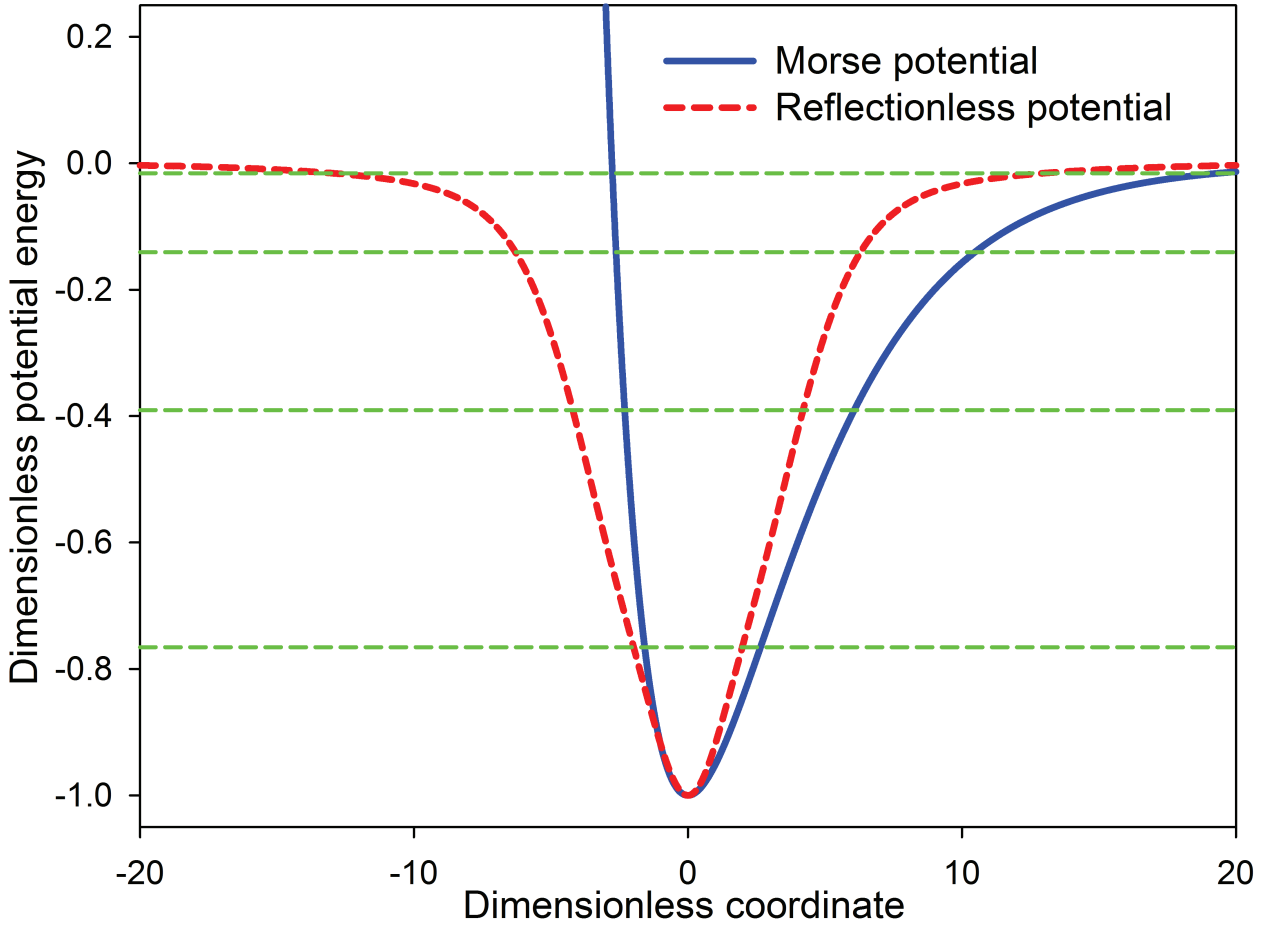


Fig. 2. The isospectral reflectionless substitute to a Morse potential (see the explanations to Example 3). The common energy levels are shown by green dashed lines.

sponds to a Morse potential [21]

$$\frac{V(x)}{D} = \exp\left(-\frac{2\alpha x}{x_0}\right) - 2\exp\left(-\frac{\alpha x}{x_0}\right), \quad (67)$$

taking  $x_0 \equiv \sqrt{\hbar^2/(2mD)} = 1$ ,  $E_0 = D = 1$ , and consequently,  $C = 1$ . The energy eigenvalues read

$$E_n = -D\left(1 - \frac{n+1/2}{a}\right)^2, \quad (68)$$

where  $a \equiv \sqrt{D/C}/\alpha$ . In Fig. 2, the value  $a = 4$  has been chosen, so that  $\alpha = 1/4$  in our dimensionless units. As in the previous example, the red dashed curve shows the symmetric reflectionless potential derived by Eq. (65) from the input parameters (66).

## 6. CONCLUSION

The main result of this work is a general formula for calculating  $\tau$ -functions. This important formula, Eq. (50), is a direct conclusion from Theorem 2 that has been proved with the help of well-known methods of the theory of determinants. We demonstrated that  $\tau$ -functions can be expanded in terms of special determinants

called alternants [16]. Any alternant has a divisor – the Vandermonde's determinant of the same order, while the quotient can be uniquely expressed as a polynomial in elementary symmetric functions (32) (see Theorem 1). Moreover, in the case of alternants related to the inverse scattering problem this quotient equals unity, i.e., the alternant itself equals the Vandermonde's determinant. These useful properties of alternants is the key to a very simple final result expressed by Eq. (50).

Using Eqs. (50)-(51), one can reconstruct any reflectionless one-dimensional potential on the full line ( $-\infty < x < \infty$ ), provided that the  $2N$  input parameters  $\kappa_n$  and  $C_n$  ( $n = 1, 2, \dots, N$ ) are known. Moreover, if the result is expected to be a symmetric function of the coordinate  $x$  then the problem can be uniquely solved on the basis of the  $N$  binding energies  $E_n = -C\kappa_n^2$ . Compared with the direct use of Eq. (9), the described approach significantly reduces computational efforts. Indeed, the expansions (21) and (50) contain only  $2^{N-1}$  members, while Eq. (9) requires evaluation of a determinant with  $N!$  members.

The efficiency of the method has been explicitly demonstrated for the case  $N = 4$ , and there is no doubt that the algorithm can be successfully applied to a much higher number (in principle, to an arbitrary number) of given binding energies. The described approach can also be applied to building  $N$ -soliton solutions to the Korteweg-de Vries equation, but this would be a subject for another paper.

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## Praktiline meetod etteantud energiaspektriga mittepeegeldava potentsiaali konstrueerimiseks

Töös kirjeldatakse puht-algebraalset meetodit, mis võimaldab konstrueerida kuitahes suure hulga ( $N$ ) etteantud diskreetsete energianivoodega mittepeegeldava potentsiaali. Konstruksiooni aluseks on üldine valem (valem (9) selles artiklis), mis taandab potentsiaali konstrueerimise uuritava süsteemi jaoks iseloomuliku  $N \times N$ -determinandi arvutamisele. Kuna taolise determinandi arendus sisaldab  $N!$  liiget, siis osutub ülesanne üsna keerukaks juba suhteliselt väikese arvu nivoodega süsteemi ( $N > 3$ ) jaoks. Seesuguste determinantide arvutamist saab aga väga oluliselt lihtsustada, kuna nad on avaldatavad spetsiifiliste omadustega determinantide – alternantide – kaudu.

Töös tõestatakse kaks üldist teoreemi ning tuletatakse lihtne ja kergesti kasutatav valem alternantide arvutamiseks. Kirjeldatud protseduuri tulemusena saadakse modifitseeritud determinant, mille arendus sisaldab ainult  $2^{N-1}$  liiget (vt valemeid (50)-(51)). Arvutustehnilises mõttes on see tohutu võit. Tõepoolest, kui näiteks  $N = 10$ , siis on arenduses  $10! = 3628800$  asemel ainult  $2^9 = 512$  liiget, ning ka märksa suurema nivoode arvu korral on arvutused tänapäevaste vahendite abil kergesti teostatavad.

Kui seada lisatingimus, et otsitav potentsiaal on reallteljel ( $-\infty, \infty$ ) sümmeetriline, lihtsustub probleem veelgi. Sel juhul on potentsiaali kuju  $N$  energianivoo kaudu üheselt määratud (üldjuhul tuleb fikseerida  $2N$  parameetrit). Meetodi tõhusust sümmeetriliste potentsiaalide konstrueerimisel on mitme näite abil ka vahetult demonstreeritud. Väärub märkimist, et töö tulemusi saab kasutada mitte ainult hajumisteooria pöördülesande kontekstis, vaid ka Korteweg-de Vries'i võrrandi  $N$ -solitoni lahendite kindlakstegemiseks.

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