Convergence of spline collocation for Volterra integral equations

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Abstract

Estimates for step-by-step interpolation projections are established. Depending on the spectrum of the transfer matrix these estimates allow to obtain the pointwise convergence of the projectors to the identity operator or, in some limit cases, to prove stable convergence of the corresponding approximate operators of integral equations. This, via general convergence theorems for operator equations, permits to get the convergence of collocation method for Volterra integral equations of the second kind in spaces of continuous or certain times continuously differentiable functions. Applications in the case of the most practical types of splines are analyzed.

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1. Introduction

We will consider the spline collocation method for the Volterra integral equation of the second kind

\[ y(t) = \int_0^t K(t, s, y(s)) \, ds + f(t), \quad t \in [0, T], \]

where \( f : [0, T] \rightarrow \mathbb{R} \) and \( K : S \times \mathbb{R} \rightarrow \mathbb{R} \) are given functions and the set \( S \) is defined by \( S = \{(t, s) : 0 \leq s \leq t \leq T\} \).

In general, we need a family of grids \( 0 = t_0 < t_1 < \cdots < t_N = T \) with \( N \rightarrow \infty \), thus the grid points \( t_n \) are dependent on \( N \). Denote by \( h_n = t_n - t_{n-1} \) the length of a particular interval \( \sigma_n = (t_{n-1}, t_n), \ n = 1, \ldots, N \), and the set of interior grid points by \( \Delta_N = \{t_1, \ldots, t_{N-1}\} \). For given integers \( m \geq 1 \) and \( d \geq -1 \) define the space of splines

\[ S_{m+d}^d = S_{m+d}^d(\Delta_N) = \{u \in C^d[0, T] : u|_{\sigma_n} \in \mathcal{P}_{m+d}, \ n = 1, \ldots, N\}, \]

where \( \mathcal{P}_k \) is the set of all polynomials of degree not exceeding \( k \).

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Let the collocation parameters \(0 < c_1 < \cdots < c_m \leq 1\) be fixed (independent of \(N\)). In order to determine the approximate solution \(u \in S_{m+d}^d\) of Eq. (1) we impose the collocation conditions

\[
u(t) = \int_0^t K(t, s, u(s))\, ds + f(t), \quad t = t_{n-1} + c_i h_n, \quad i = 1, \ldots, m, \quad n = 1, \ldots, N.\]  

(2)

A principal extension of the treatment is the collocation with multiple knots which we allow as well. Then the number of different parameters \(c_i\) is less than \(m\) and in a knot \(t_{n-1} + c_i h_n\), having multiplicity \(m_i\), we impose the conditions

\[
u^{(j)}(t) = \frac{d^j}{dt^j} \int_0^t K(t, s, u(s))\, ds + f^{(j)}(t), \quad t = t_{n-1} + c_i h_n, \quad j = 0, \ldots, m_i - 1.\]  

(3)

Assuming that Eq. (1) has the solution \(y \in C^k = C^k[0, T]\) for some \(k \geq 0\) it is quite natural to impose initial conditions \(u^{(j)}(0) = y^{(j)}(0)\) for \(j = 0, \ldots, \min(k, d)\), and \(u^{(j)}(0) = 0\) for \(j = k + 1, \ldots, d\) (in case \(k < d\)). Of course, instead of conditions on derivatives at 0, additional collocation points on the first interval or even “not-a-knot” condition at \(t_1\) could be used to start the step-by-step determination of \(u\). To determine on any next interval the approximate solution \(u\), having \(m + d + 1\) free parameters as element of \(\mathcal{P}_{m+d}\), we use \(m\) collocation and \(d + 1\) smoothness conditions.

The convergence problem of step-by-step collocation with polynomial splines of arbitrary degree and arbitrary smoothness for Volterra integral equations of the second kind has been open for many years. In the particular case of ordinary differential equations spline collocation methods, where instead of (2) and (3) the approximate solution \(u \in S_{m+d}^d\) satisfies the equation \(y'(t) = f(t, y(t))\) at the collocation points and appropriate initial conditions, have been studied for a long time. Loscalzo and Talbot [15] showed that collocation with smooth splines \((m = 1, d \geq 0, c_1 = 1)\) is convergent for \(m \leq 3\) and divergent for \(m \geq 4\). In [10,25] it is shown that collocation with continuous splines without any smoothness \((d = 0)\) is equivalent to implicit Runge–Kutta methods and is convergent for any values of collocation parameters \(c_i, i = 1, \ldots, m\). Callender [7] showed that collocation with continuously differentiable splines \((d = 1)\) and uniformly spaced collocation points is convergent for any \(m \geq 1\). Quite complete convergence and divergence theory, including multiple collocation points, is established by Mühlthi [16,17], a thorough survey and many additional references can be found in [3]. There are also several successful attempts to generalize the ideas working in the case of differential equations to Volterra integral equations. Probably the most widely used in practice and theoretically studied class of methods for initial value problems of ordinary differential equations is Runge–Kutta methods. A comprehensive generalization for Volterra integral equations could be found in [4], see also [5]. First important developments in this direction are presented already by Pouzet [23] and Bel’tjukov [2]. Runge–Kutta methods are fully discretized and give a finite number of approximate values (at grid points) to the exact solution. Spline collocation for general equations requires exact evaluation of certain integrals but gives a function as an approximate solution which is a principal advantage compared to Runge–Kutta methods. Let us mention that, with suitable evaluation of integrals by interpolatory quadratures, in some special cases, the spline collocation is equivalent to Runge–Kutta methods, for details see [5]. We refer the reader to [5] for an historical survey of results obtained until the middle of 80s. A particular case of spline collocation methods for solving Volterra integral equations is the piecewise polynomial collocation method \((d = -1)\), which is known to be always convergent. A comprehensive coverage of convergence results of such methods with an extensive annotated list of references can be found in [3]. We also refer to [6], where a wide class of nonlinear Volterra integral equations (including equations with weakly singular kernels) is considered. Convergence and the rate of convergence of Hermite cubic spline collocation for Volterra integral equations with a special weakly singular kernel is investigated in [8]. Results about convergence of collocation using the same splines for Volterra integro-differential equations with weakly singular kernels are obtained in [22]. On the other hand, it is known [12,18,19] that the step-by-step spline collocation with smooth splines (cubic and higher order i.e. \(d \leq 2, m = 1\)) is always unstable. An approach which uses different projections is proposed in [20,21]. This method requires solution of a system of equations, but has still the same complexity and order of convergence as step-by-step methods.

In this paper we prove the convergence and convergence rates of the step-by-step spline collocation methods for Volterra integral equations under purely algebraic conditions depending only on the smoothness and the order of the
splines and the collocation parameters (see Theorems 8 and 14). This is achieved by characterizing first the approximation properties of the corresponding step-by-step spline interpolation operators in terms of the eigenvalues of a transfer matrix (see Theorem 1) and, using the obtained results, demonstrating that certain general convergence theorems can be applied in studying the collocation methods for Volterra integral equations. In certain cases (when the eigenvalues of the transfer matrix are strictly inside the unit circle) the interpolation operators converge strongly to the identity operator in the appropriate function spaces and application of the general convergence theory is straightforward. If all eigenvalues of the transfer matrix are in the closed unit disk and those which are on the unit circle have equal algebraic and geometric multiplicities, then approximating operators itself are not good enough but they behave well together with the Volterra integral operators. In the last case it is not trivial to show that the general convergence theory is applicable.

It follows from Theorem 1 that our convergence theory is quite complete: if the conditions of the convergence theorems are not satisfied, then the step-by-step interpolation process is in general exponentially unstable and therefore there is no hope to obtain convergence results for any class of Volterra integral equations containing the case where the kernel of the integral operator is constantly zero.

Since the algebraic conditions of the convergence theorems involve eigenvalues of a certain matrix it is natural to ask if it is possible to reduce the conditions to explicit relations between the order and smoothness of the spline space and the collocation parameters. This is at the moment an open problem but in Section 6 we demonstrate that for many values of $m$ and $d$ it is possible to get explicit relations for the collocation parameters that guarantee the applicability of the convergence theorems.

2. Estimates for interpolation operators

Define interpolation operators $P_N : C^k \rightarrow S_{m+d}^d$ such that, for $f \in C^k$, we have $P_N f \in S_{m+d}^d$ and

$$
(P_N f)^{(j)}(0) = f^{(j)}(0), \quad j = 0, \ldots, \min(k, d),
$$

$$
(P_N f)^{(j)}(0) = 0, \quad j = k + 1, \ldots, d, \text{ if } k < d,
$$

$$
(P_N f)(t_{n-1} + c_i h_n) = f(t_{n-1} + c_i h_n), \quad i = 1, \ldots, m, \ n = 1, \ldots, N,
$$

together with

$$
(P_N f)^{(j)}(t_{n-1} + c_i h_n) = f^{(j)}(t_{n-1} + c_i h_n), \quad j = 1, \ldots, m_i - 1,
$$

if $t_{n-1} + c_i h_n$ has multiplicity $m_i \geq 2$.

However, other initial conditions considered above could be used. Let us note that, for $k \leq d$, we have $S_{m+d}^d \subset C^k$ and the operators $P_N$ could be considered as projectors in the space $C^k$.

Consider the integral operator defined by

$$
(K_y)(t) = \int_0^t K(t, s, y(s)) \, ds, \ t \in [0, T].
$$

It is a standard calculation to check that the method which uses collocation conditions (2) or (3) and initial conditions consistent with the definition of $P_N$ is equivalent to the equation

$$
u = P_N K u + P_N f, \quad u \in S_{m+d}^d.
$$

In Section 3 we present some general convergence theorems and later on apply them to Eqs. (1) and (5).

Our main results will be established for the uniform mesh, thus, let us take now $h_n = h = T/N, \ t_n = nh, \ n = 0, \ldots, N$.

Assuming $f \in C^k$, we have $P_N f \in S_{m+d}^d$. On the interval $(t_{n-1}, t_n)$ we use the representation

$$
(P_N f)(t_{n-1} + \tau h) = \sum_{j=0}^{m+d} \alpha_{nj} \tau^j, \quad \tau \in (0, 1).
$$
The requirement $P_N f \in C^d$ gives the conditions
\[
\alpha_{n+1,j} = \sum_{\ell=j}^{m+d} \binom{\ell}{j} \alpha_{n,\ell}, \quad j = 0, \ldots, d, \quad n = 1, \ldots, N-1. \tag{6}
\]

Interpolation conditions $(P_N f)(t_n + c_i h) = f(t_n + c_i h)$ lead to the equations
\[
\sum_{\ell=0}^{m+d} \alpha_{n+1,\ell} c_i^\ell = f(t_n + c_i h), \quad i = 1, \ldots, m, \quad n = 0, \ldots, N-1. \tag{7}
\]

In the multiple knot case, the condition $(P_N f)^{(j)}(t_n + c_i h) = f^{(j)}(t_n + c_i h)$ could be written as
\[
\sum_{\ell=j}^{m+d} \alpha_{n+1,\ell} \binom{\ell}{j} c_i^{\ell-j} = \frac{h^j}{j!} f^{(j)}(t_n + c_i h). \tag{8}
\]

Eqs. (6) and (7) can be written in the matrix form as
\[
V \alpha_{n+1} = V_0 \alpha_n + f_n, \quad n = 1, \ldots, N-1, \tag{9}
\]
where $\alpha_n = (\alpha_{n0}, \ldots, \alpha_{nm+d})$, $V$ and $V_0$ are $(m + d + 1) \times (m + d + 1)$ matrices
\[
V = \left( \frac{I}{C} \right), \quad V_0 = \left( \frac{A}{0} \right).
\]
$I$ is the $(d + 1) \times (d + 1)$ identity matrix, $C$ has $m$ rows in the form $(1 c_1 \ldots c_i^{m+d})$, $A$ is a $(d + 1) \times (m + d + 1)$ matrix with elements
\[
\alpha_{\ell j} = \begin{cases} 0, & \ell < j, \\ \binom{\ell}{j}, & \ell \geq j, \quad \ell = 0, \ldots, d, \quad j = 0, \ldots, m + d, \end{cases}
\]
and $f_n$ is a $(m + d + 1)$-dimensional vector $f_n = (f_{n0}, \ldots, f_{nm+d})$ with components
\[
f_{nj} = \begin{cases} 0, & j \leq d, \\ f(t_n + c_{j-d} h), & j > d, \end{cases}
\]
with obvious modifications in $C$ and $f_n$ prescribed by (8) in case of multiple knots. In the sequel we present detailed analysis only for the case (7), not pointing out the details for (8).

Clearly, the matrix $V$ is invertible and (9) can be written equivalently in the form
\[
\alpha_{n+1} = M \alpha_n + V^{-1} f_n, \quad n = 1, \ldots, N-1, \tag{10}
\]
where $M = V^{-1} V_0$.

Denote by $\tau(M)$ the spectral radius of $M$. For $u \in S_{m+d}^d$, let $u^{(\ell)}$ denote the $\ell$-th derivative on each subinterval $(t_{n-1}, t_n)$ separately. We need also the modulus of continuity
\[
\omega(g, h) = \sup_{|x-y| \leq h, x, y \in [0, T]} |g(x) - g(y)|.
\]

Now we are ready to state an approximation result.

**Theorem 1.** For $\tau(M) < 1$, we have
\[
\|(P_N f)^{(\ell)} - f^{(\ell)}\|_\infty \leq \text{const} h^{k-\ell} \omega(f^{(k)}, h), \quad 0 \leq \ell \leq k \leq m + d. \tag{11}
\]

For $\tau(M) = 1$, it holds
\[
\|(P_N f)^{(\ell)} - f^{(\ell)}\|_\infty \leq \text{const} T^r h^{k-\ell-r} \omega(f^{(k)}, h), \quad 0 \leq \ell \leq k \leq m + d, \tag{12}
\]
where $r$ is the maximal dimension of Jordan blocks of $M$ corresponding to eigenvalues $\lambda_i$ such that $|\lambda_i| = 1$.

If $\tau(M) > 1$, then there exists a function $f \in C^{m+d}$ such that $P_N f$ does not converge to $f$ in $C[0, T]$.

Constants occurring in (11) and (12) may depend on $m$, $d$ and $c_i$ but not on $h$ and $T$. 
Proof. We will estimate $P_N f - f$ and $(P_N f)^{(\ell)} - f^{(\ell)}$ in uniform norm on subintervals $(t_{n-1}, t_n)$. From

$$(P_N f)(t_{n-1} + \tau h) - f(t_{n-1} + \tau h) = \sum_{j=0}^{m+d} \alpha_{nj} \tau^j - \sum_{j=0}^{k} \frac{h^j}{j!} f^{(j)}(t_{n-1}) \tau^j + \frac{h^k}{k!} f^{(k)}(t_{n-1}) \tau^k - \frac{h^k}{k!} f^{(k)}(\xi_n) \tau^k$$

where $\xi_n \in (t_{n-1}, t_n - \tau h)$

$$\beta_{nj} = \begin{cases} \frac{h^j}{j!} f^{(j)}(t_{n-1}), & 0 \leq j \leq k, \\ 0, & k < j \leq m + d, \end{cases}$$

with $\beta_n = (\beta_{n0}, \ldots, \beta_{nm+d})$ we get

$$\left\| (P_N f)(t_{n-1} + \tau h) - f(t_{n-1} + \tau h) \right\| \leq \text{const} \left( \|\alpha_n - \beta_n\|_\infty + h^k \omega(f^{(k)}, h) \right).$$

(13)

Similarly, the expansion

$$(P_N f)^{(\ell)}(t_{n-1} + \tau h) - f^{(\ell)}(t_{n-1} + \tau h) = \frac{1}{h^\ell} \sum_{j=\ell}^{m+d} \frac{j!}{(j - \ell)!} \alpha_{nj} \tau^{j-\ell} - \sum_{j=\ell}^{k} \frac{h^{j-\ell}}{(j - \ell)!} f^{(j)}(t_{n-1}) \tau^{j-\ell}$$

$$+ \frac{h^{k-\ell}}{(k - \ell)!} f^{(k)}(t_{n-1}) \tau^{k-\ell} - \frac{h^{k-\ell}}{(k - \ell)!} f^{(k)}(\xi_n) \tau^{k-\ell}$$

yields

$$\left\| (P_N f)^{(\ell)}(t_{n-1} + \tau h) - f^{(\ell)}(t_{n-1} + \tau h) \right\| \leq \text{const} (h^{-\ell} \|\alpha_n - \beta_n\|_\infty + h^{k-\ell} \omega(f^{(k)}, h)).$$

(14)

Consider the equation

$$V(\alpha_{n+1} - \beta_{n+1}) = V_0(\alpha_n - \beta_n) + r_n.$$  

(15)

For $j = 0, \ldots, k$ and $n = 1, \ldots, N - 1$, we have

$$\beta_{n+1,j} - \sum_{\ell=j}^{m+d} \binom{\ell}{j} \beta_{n,\ell} = \frac{h^j}{j!} \left( f^{(j)}(t_n) - \sum_{\ell=j}^{k} \frac{h^{j-\ell}}{(\ell - j)!} f^{(\ell)}(t_{n-1}) \right)$$

$$= \frac{h^k}{j!(k-j)!} (f^{(k)}(\xi_{nj}) - f^{(k)}(t_{n-1}))$$

with $\xi_{nj} \in (t_{n-1}, t_n)$ and hence

$$\left| \beta_{n+1,j} - \sum_{\ell=j}^{m+d} \binom{\ell}{j} \beta_{n,\ell} \right| \leq \text{const} h^k \omega(f^{(k)}, h).$$

(16)

Similarly

$$\left| f(t_n + c_i h) - \sum_{\ell=0}^{m+d} c_i^\ell \beta_{n+1,\ell} \right| = \left| f(t_n + c_i h) - \sum_{\ell=0}^{k} \frac{f^{(\ell)}(t_n)(c_i h)^\ell}{\ell!} \right|$$

$$\leq \text{const} h^k \omega(f^{(k)}, h), \quad i = 1, \ldots, m, \quad n = 0, \ldots, N - 1.$$  

(17)

Based on (16) and (17) we have $\|r_n\|_\infty \leq \text{const} h^k \omega(f^{(k)}, h)$.

The relation (15) permits us to get

$$\alpha_{n+1} - \beta_{n+1} = M^n(\alpha_1 - \beta_1) + \sum_{j=0}^{n-1} M^j V^{-1} r_{n-j}, \quad n = 1, \ldots, N - 1.$$  

(18)
First, let us analyze the term $\alpha_1 - \beta_1$. Note that the first $d + 1$ components of $\alpha_1$ and $\beta_1$ coincide. Taking into account (7) we have

$$
\sum_{\ell=d+1}^{m+d} (\alpha_1 \ell - \beta_1 \ell) c_\ell = \sum_{\ell=0}^{m+d} (\alpha_1 \ell - \beta_1 \ell) c_\ell = f(t_0 + c_1 h) - \sum_{\ell=0}^{m+d} \beta_1 c_\ell. $

The use of (17) allows to estimate the right-hand side of this system and, consequently, the solution too. Thus, last $m$ components of $\alpha_1 - \beta_1$ could be estimated by $\text{const} h^k \omega(f(k), h)$.

It is well known that, for any $\varepsilon > 0$, one may choose a norm in $\mathbb{R}^{m+d+1}$ such that $\|M\| \leq r(M) + \varepsilon$. Thus, if $r(M) < 1$, for sufficiently small $\varepsilon$, we have $\|M\| < 1$. Estimating the left-hand side of (18) in this vector norm we get $\|\alpha_n - \beta_n\|_{\infty} \leq \text{const} h^k \omega(f(k), h)$, which, in turn, due to (13) and (14) gives the first assertion of theorem.

Next we apply the following fact: there exists a vector norm such that the corresponding matrix norm is equal to the spectral radius of the matrix if and only if all eigenvalues with maximal modulus have equal algebraic and geometric multiplicities. If $M$ satisfies this condition and $r(M) = 1$, we may consider a norm in $\mathbb{R}^{m+d+1}$ such that $\|M\| = 1$. Then, taking into account that $N = T/h$ and $h \leq T$, (18) leads to

$$
\|\alpha_{n+1} - \beta_{n+1}\|_{\infty} \leq \text{const} h^k \omega(f(k), h) + \text{const} N h^k \omega(f(k), h) \leq \text{const} T h^{k-1} \omega(f(k), h).
$$

This proves the second assertion of theorem in the case $r = 1$.

Suppose now that $r(M) = 1$ and there is at least one eigenvalue $\lambda_i$ of $M$ such that $|\lambda_i| = 1$ with different algebraic and geometric multiplicities.

Take a spectral basis $v_{jq}$, $j = 1, \ldots, p$, $q = 1, \ldots, n_j$, in $\mathbb{R}^{m+d+1}$ such that $M v_{j1} = \lambda_j v_{j1}$, $M v_{jq} = \lambda_j v_{jq} + v_{j,q-1}$, $q = 2, \ldots, n_j$, $j = 1, \ldots, p$. Actually, this means also that the matrix $M$ has $p$ Jordan blocks and $j$-th of them has dimension $n_j$. Any $v \in \mathbb{R}^{m+d+1}$ can be uniquely represented in the form $v = \sum_{j=1}^{p} \sum_{q=1}^{n_j} c_{jq} v_{jq}$ and

$$
\|v\| = \max_{j=1,\ldots,p, q=1,\ldots,n_j} |c_{jq}|
$$

is a norm in $\mathbb{R}^{m+d+1}$. A straightforward calculation gives

$$
M^i v = \sum_{j=1}^{p} \sum_{q=1}^{n_j} \sum_{\ell=1}^{\min(n_j,i+q)} c_{jq} \left( \frac{i}{\ell - q} \right)^{\lambda_j^{i-(\ell-q)}} v_{jq}.
$$

Hence

$$
\|M^i v\| \leq \max_{j,q} |c_{jq}| \left( \sum_{j=1}^{p} \sum_{q=1}^{n_j} \sum_{\ell=1}^{\min(n_j,i+q)} \left( \frac{i}{\ell - q} \right)^{|\lambda_j|^{i-(\ell-q)}} \right) \leq \sum_{j=1}^{p} \sum_{q=1}^{n_j} \left( \frac{i}{\ell - 1} \right)^{|\lambda_j|^{i-(\ell-1)}} \|v\|.
$$

This allows to get, for $i \leq \max_{j=1,\ldots,p} n_j$, that $\|M^i\| \leq \text{const}$, otherwise

$$
\|M^i\| \leq \text{const} \sum_{j=1}^{p} n_j^2 n_j^{-1} |\lambda_j|^{i-(n_j-1)}.
$$

In this sum, the terms with $|\lambda_j| < 1$ remain bounded as $i$ increases. The same could be observed for those with $|\lambda_j| = 1$ and $n_j = 1$. Taking also into account the rest, we get

$$
\|M^i\| \leq \text{const} N^{r-1}, \quad i = 0, \ldots, N,
$$

where $r = \max\{n_j : |\lambda_j| = 1\}$. Applying (19) in (18) we obtain the second assertion of theorem.
Let us consider now the case \( r(M) > 1 \). Note that this is not possible for \( d = -1 \) since in that case we have \( M = 0 \). Let \( \lambda \) be an eigenvalue of \( M \) such that \( |\lambda| > 1 \) and let \( v = (v_0, \ldots, v_{m+d}) \) be a corresponding nonzero eigenvector. Choose a function \( g \in C^{m+d}[0, \infty) \) such that \( g^{(j)}(0) = j!v_j, \ j = 0, \ldots, d, \) and \( g(t) = 0, t \geq c_1 \). Define
\[
g_N(t) = h^{m+d} g\left(\frac{t}{h}\right), \quad t \in [0, T],
\]
then the functions \( g_N \) are bounded in \( C^{m+d}[0, T] \). The initial conditions
\[
(P_N g_N)^{(j)}(0) = g_N^{(j)}(0), \quad j = 0, \ldots, d,
\]
give
\[
a_{\alpha j} = h^{m+d}v_j, \quad j = 0, \ldots, d.
\]
In addition, the interpolation conditions \( P_N g_N(c_i h) = 0, i = 1, \ldots, m, \) and the last \( m \) equations in the system \( V_0 v = \lambda V v \) (equivalent to \( M v = \lambda v \)) are the same homogeneous equations for \( \alpha \) and \( v \) which yield \( \alpha_1 = h^{m+d}v \). Thus we have
\[
(P_N g_N)(t_{N-1} + \tau h) = \lambda^{N-1} h^{m+d} \sum_{j=0}^{m+d} v_j \tau^j
\]
and the operators \( P_N \) are not uniformly bounded as operators from \( C^{m+d}[0, T] \) to \( C[0, T] \). Hence, according to Banach–Steinhaus theorem, there exists a function \( f \in C^{m+d} \) such that \( P_N f \) does not converge to \( f \) in \( C[0, T] \). The proof is complete. \( \square \)

**Remark 2.** Using the local error estimates of \([9,13]\) it is possible to get by repeating the arguments of the proof of Theorem 1 analogs of the estimates (11) and (12) for more general classes of functions \( f \).

**Remark 3.** The estimates (11) and (12) remain valid if we extend the definition of \( P_N \) to the functions \( f \in C[0, T] \) such that \( f \in C^k[0, h] \), for \( n = 1, \ldots, N - 1 \) and \( \ell = 1, \ldots, k, f^{(\ell)} \in C(nh, (n+1)h) \), \( f^{(\ell)} \) is extendable to \( f^{(\ell)} \in C[nh, (n+1)h] \) and, instead of \( \omega(f^{(k)}, h) \) we use the maximal oscillation on subintervals
\[
\omega_N(f^{(k)}) = \max_{1 \leq n \leq N} \sup_{x, y \in (t_{n-1}, t_n)} |f^{(k)}(x) - f^{(k)}(y)|.
\]

### 3. General convergence theorems

In this section we present some general convergence theorems for operator equations which will be applied to Volterra integral equations (1). We start by recalling some notions and refer the reader to \([24]\) for more detailed presentation.

Let \( E \) and \( E_n, \ n = 1, 2, \ldots, \) be Banach spaces. Let a sequence of connecting operators \( p_n \in \mathcal{L}(E, E_n) \) with the property
\[
\|p_n x\| \to \|x\| \quad \forall x \in E
\]
as \( n \to \infty \) be given. A sequence \( x_n \in E_n \) is said to be convergent to \( x \in E \) (written as \( x_n \to x \)) if \( \|x_n - p_n x\| \to 0 \).

The sequence \( x_n \in E_n \) is compact if for any infinite subset \( N' \subset \mathbb{N} \) there is infinite \( N'' \subset N' \) and \( x \in E \) such that \( x_n \to x, n \in N'' \). A sequence \( A_n \in \mathcal{L}(E_n, E_n) \) is said to be convergent to \( A \in \mathcal{L}(E, E) \) and this is denoted by \( A_n \to A \) if \( x_n \to x \) implies \( A_n x_n \to A x \). The convergence \( A_n \to A \) is compact if \( \|x_n\| \leq \text{const} \) implies the compactness of \( A_n x_n \). The convergence \( A_n \to A \) is stable if, for \( n \geq n_0 \) with some \( n_0 \), it holds \( A_n^{-1} \in \mathcal{L}(E_n, E_n) \) and \( \|A_n^{-1}\| \leq \text{const} \).

The convergence \( A_n \to A \) is regular if the conditions \( \|x_n\| \leq \text{const} \) and \( A_n x_n \) compact imply that \( x_n \) is compact.

Consider the equation
\[
u = Ku + f
\]
with a compact linear operator \( K \in \mathcal{K}(E, E) \) and \( f \in E \). Let a sequence of operators \( K_n \in \mathcal{K}(E_n, E_n) \), elements \( f_n \in E_n \) and equations
\[
u_n = K_n \nu_n + f_n
\]
be also given. We will use the following version of a more general theorem, see [24].

**Theorem 4.** Suppose \( u = Ku \) only if \( u = 0 \), \( I_n - K_n \rightarrow I - K \) stably or regularly and \( f_n \rightarrow f \).

Then

1. (21) has a unique solution \( u \);
2. there is \( n_0 \) such that, for \( n \geq n_0 \), (22) has a unique solution \( u_n \);
3. \( u_n \rightarrow u \);
4. there are positive constants \( C_1, C_2 \) such that
   \[
   C_1 \| (I_n - K_n)p_n u - f_n \| \leq \| u_n - p_n u \| \leq C_2 \| (I_n - K_n)p_n u - f_n \|. \tag{23}
   \]

Actually, from Theorem 4 follows

**Theorem 5.** Suppose that the assumptions of Theorem 4 are satisfied with the exception that, instead of stable or regular convergence of operators, we have \( K_n \rightarrow K \) compactly. Then the assertions (i)–(iv) hold.

An important particular case of this general approximation framework is when \( E_n = E \) for all \( n \) and the connecting operators \( p_n \) are identity operators. Let a sequence of operators \( P_n \in \mathcal{L}(E,E) \) and equations

\[
\begin{align*}
    u_n &= P_n Ku_n + P_n f. \tag{24}
\end{align*}
\]

be given. Then, being a corollary of Theorem 5, it holds

**Theorem 6.** Suppose \( u = Ku \) only if \( u = 0 \) and \( P_n u \rightarrow u \) for all \( u \in E \) as \( n \rightarrow \infty \). Then the assertions (i), (ii) for (24) and (iii) of Theorem 4 hold and instead of (23) we have with positive constants \( C_1, C_2, C_3 \) the estimates

\[
\begin{align*}
    C_1 \| P_n u - u \| &\leq \| u_n - u \| \leq C_2 \| P_n u - u \|, \\
    \| u_n - P_n u \| &\leq C_3 \| K(P_n u - u) \|. \tag{25}
\end{align*}
\]

\[
\begin{align*}
    \| u_n - P_n u \| &\leq C_3 \| K(P_n u - u) \|. \tag{26}
\end{align*}
\]

Let us note that Theorem 6 is one of the most important tools in the numerical treatment of integral equations (see [1,11]).

**Remark 7.** A modification of Theorem 4 for the approximating equations (24), where we suppose \( I - P_n K \rightarrow I - K \) stably or regularly and \( P_n f \rightarrow f \), allows to establish the estimate (25) instead of (23).

4. **Convergence results for integral equations**

Consider the linear Volterra integral equation

\[
y(t) = \int_0^t K(t,s)y(s)\,ds + f(t), \quad t \in [0, T],
\]

with given functions \( f : [0, T] \rightarrow \mathbb{R} \) and \( K : S \rightarrow \mathbb{R} \). Denote by \( K \) the integral operator defined by

\[
(Ky)(t) = \int_0^t K(t,s)y(s)\,ds, \quad t \in [0, T].
\]

In Section 2 we introduced interpolation operators \( P_N : C^k \rightarrow S_{m+d}^d \) and we saw that the spline collocation method is equivalent to a linear equation in space \( S_{m+d}^d \)

\[
\begin{align*}
    u_N &= P_N Ku_N + P_N f. \tag{28}
\end{align*}
\]

In this section we give two convergence theorems for the case \( r(M) < 1 \). The case \( r(M) = 1 \) with \( r = 1 \) will be considered in Section 5. Let us note that \( r(M) > 1 \) causes exponential instability of the collocation method even
for constant kernel in (27). To see that, one should take into account the first reasoning in Section 6 connecting the transfer matrix $M$ with those of [18] and then look at [18]. Although in [18] the exponential instability is proved only for collocation points with multiplicity one, the technique elaborated therein works as well in the general case.

**Theorem 8.** Suppose the kernel in (27) is such that $K \in \mathcal{K}(C^k, C^k)$, $u = Ku$ only if $u = 0$ and $f \in C^k$. Let the matrix $M$ in (10) have $\tau(M) < 1$ and $k \leq d$.

Then (27) has unique solution $y$, for sufficiently large $N$ Eq. (28) has unique solution $u_N$, the sequence $u_N$ converges to $y$ in the space $C^k$ as $N \to \infty$ with the estimates

$$
C_1 \|P_N y - y\| \leq \|u_N - y\| \leq C_2 \|P_N y - y\|,
$$

$$
\|u_N - P_N y\| \leq C_3 \|K(P_N y - y)\|.
$$

Proof. is an immediate application of Theorem 6 if we take $E = C^k$ and, basing on (11), observe that $P_N \to I$.

**Remark 9.** Let us mention without details that the convergence $P_N \to I$ allows to establish a counterpart of Theorem 8, including two-sided error estimates, for nonlinear equation (1). One needs to require only that the operator $K$ is continuous and compact, see [14].

**Theorem 10.** Suppose $K \in C^k(S)$ and $f \in C^k$. Let the matrix $M$ in (10) be such that $\tau(M) < 1$ and $k > d$. Then the assertions of Theorem 8 about solvability of (27) and (28) hold, $u_N$ converges to $y$ with the estimate

$$
\|u_N - y\|_k \leq \text{const} \|P_N y - y\|_k
$$

where $\|u\|_k = \sum_{\ell=0}^k \max_{0 \leq \ell \leq N} \sup_{t_{\ell-1} < t \leq t_\ell} |u^{(\ell)}(t)|$.

Proof. Note that already $K \in C(S)$ guarantees unique solvability of (27).

We intend to apply Theorems 4 and 5. Take $E = C^k$, $E_N = S_{m+d}$ with norms $\|u\|_E = \sum_{\ell=0}^k \max_{0 \leq \ell \leq T} \|u^{(\ell)}(t)\|$ and $\|u\|_{E_N} = \|u\|_k$ respectively. Let us take $P_N$ to be the connecting operators needed in the approximation framework. Then (11) gives that, for all $u \in E$, $P_N u \to u$. For $u_N \in E_N$ and $u \in E$, the convergence $u_N \to u$ means that $\|u_N - P_N u\| \to 0$, which is equivalent to $\|u_N - u\|_k \to 0$.

Let $u_N \to u$. Then

$$
\|P_N Ku_N - P_N Ku\|_k \leq \|P_N\|_k \|K\|_k \|u_N - u\|_k \to 0
$$

and, thus, $P_N K \to K$.

Assume first, that $k - d = 1$. Let us show that then the convergence $P_N K \to K$ is compact.

Suppose $\|u_N\|_k \leq \text{const}$. For $k = 0$ and $d = -1$, the sequence $Ku_N$, being uniformly bounded and equicontinuous, is compact in $C$. For $k \geq 1$, the sequence $u_N$ is compact in $C^{k-1}$ which, in turn, gives that $Ku_N$ is compact in $C^{k}$. To see the last, let us look at

$$
(Ku_N)^{(\ell)}(t) = \int_0^t \frac{\partial \ell \mathcal{K}}{\partial t^{\ell}}(t,s)u_N(s) \, ds + \sum_{j=0}^{\ell-1} g_{\ell j}(t)u_N^{(j)}(t), \quad \ell = 0, \ldots, k,
$$

where $g_{\ell j}(t)$ are linear combinations of derivatives of $\mathcal{K}$ until order $\ell - 1$ on the diagonal $t = s$, and conclude the compactness of $(Ku_N)^{(\ell)}$, $1 \leq \ell \leq k$, in $C$. Taking any subsequence $N' \subset N$, there are $N'' \subset N'$ and $v \in C^k$ such that $Ku_N \to v$ in $C^k$ and, consequently, $P_N Ku_N \to v$ as $N \to N''$.

Next, let $k$ and $d$ be arbitrary with $k - d \geq 2$. We show that $I - P_N K \to I - K$ regularly.

Suppose $\|u_N\|_k \leq \text{const}$ and $(I - P_N K)u_N$ is compact. We have to show that $u_N$ is compact.

Take any subsequence $N' \subset N$. Then there are $N'_0 \subset N'$ and $v \in C^k$ such that

$$
\|(I - P_N K)u_N - v\|_k \to 0, \quad N \in N'_0.
$$

Again, as in the case $k - d = 1$, for $d = -1$, the sequence $Ku_N$ is compact in $C$. For $d \geq 0$, we obtain the compactness of $Ku_N$ in $C^{d+1}$. Choose $N'' \subset N'_0$ and $w \in C^{d+1}$ such that

$$
\|Ku_N - w\|_{C^{d+1}} \to 0, \quad n \in N''.
$$
Then
\[ \| P_N Ku_N - w \|_{d+1} \to 0, \quad n \in \mathbb{N}^\prime, \]
and
\[ \| u_N - (v + w) \|_{d+1} \leq \| (I - P_N K)u_N - v \|_{d+1} + \| P_N Ku_N - w \|_{d+1} \to 0, \quad N \in \mathbb{N}^\prime. \]
Denote \( u = v + w \). As \( v, w \in C^{d+1}, u \in C^{d+1} \) and \( Ku \in C^{d+2} \), Continuing with the general step, let \( d + 2 \leq \ell \leq k \) and \( \| u_N - u \|_{\ell-1} \to 0 \) with \( u \in C^{\ell-1}, Ku \in C^{\ell} \) and
\[ \| (P_N Ku_N)(\ell) - w(\ell) \|_0 \to 0, \quad N \subset \mathbb{N}^\prime. \] (30)
Using (29) we establish the convergence
\[ \| (Ku_N)(\ell) - (Ku)(\ell) \|_0 \to 0, \quad N \in \mathbb{N}^\prime. \]
Then, taking into account Remark 3, we have
\[ \| (P_N Ku_N)(\ell) - (Ku_N)(\ell) \|_0 \leq \text{const} \omega_N((Ku_N)(\ell)) \to 0, \quad N \in \mathbb{N}^\prime, \]
and
\[ \| (P_N Ku_N)(\ell) - (Ku)(\ell) \|_0 \to 0, \quad N \in \mathbb{N}^\prime. \]
Similarly
\[ \| (P_N Ku_N)(\ell-1) - (Ku)(\ell-1) \|_0 \to 0, \quad N \in \mathbb{N}^\prime. \] (31)
Actually, (31) was obtained already on the previous step of induction except the first one with \( \ell = d + 2 \). Since by (30) and (31) we have \( w(\ell-1) = (Ku)(\ell-1) \), it now follows that \( w(\ell) = (Ku)(\ell) \) and hence \( w \in C^{\ell}. \) Now get
\[ \| u_N^{(\ell)} - (v + w)^{(\ell)} \|_0 \leq \| (I - P_N K)u_N^{(\ell)} - v^{(\ell)} \|_0 + \| (P_N Ku_N)^{(\ell)} - w^{(\ell)} \|_0 \to 0, \quad N \in \mathbb{N}^\prime, \]
thus \( \| u_N - u \|_{\ell} \to 0, \quad N \in \mathbb{N}^\prime. \)
Finally, the error estimate of Theorem 10 follows from (23). The proof is complete. \( \square \)

**Remark 11.** In general, the convergence \( P_N K \to K \) is not compact. For example, take \( (Ku)(t) = \int_0^t u(s) \, ds, k = 1 \) and \( d = -1 \). Choose \( u_N \) such that \( u_N(t) = (-1)^{n-1}, \quad t \in (t_{n-1}, t_n), \quad n = 1, \ldots , N \). Then we have
\[ (Ku_N)(t_n) = \begin{cases} 0, & \text{for } n \text{ even}, \\ h, & \text{for } n \text{ odd}, \end{cases} \]
and \( (Ku_N)'(0) = u_N'(0) = 1 \). The assumption that there are \( v \in C^1 \) and a subsequence of \( P_N Ku_N \) converging to \( v \) in 1-norm leads to the contradiction.

Theorems 1, 8 and 10 enable us to get easily various global convergence and local superconvergence (at the collocation points) results. As an example, we have

**Corollary 12.** Assume that \( K \in C^{m+d+1}(S) \) and \( f \in C^{m+d+1}[0, T] \) for some \( d \geq -1 \) and \( m \geq 1 \). Let the matrix \( M \) in (10) be such that \( r(M) < 1 \). Then (27) has a unique solution \( y \), for sufficiently large \( N \) Eq. (28) has a unique solution \( u_N \) and the error estimate
\[ \| u_N - y \|_{\infty} \leq \text{const} h^{m+d+1} \] (32)
holds.

**Proof.** Using either Theorem 8 (if \( d \geq 0 \)) or Theorem 10 (if \( d = -1 \)) with \( k = 0 \) we get
\[ \| u_N - y \|_{\infty} \leq \text{const} \| y - P_N y \|_{\infty}. \]
Under the assumptions of the corollary we have \( y \in C^{m+d+1}[0, T] \), hence the estimate (11) of Theorem 1 (with \( k = m + d, \ell = 0 \)) gives us
\[ \| y - P_N y \|_{\infty} \leq \text{const} h^{m+d} \omega(y^{(m+d)}, h) \leq \text{const} h^{m+d+1}. \]
This concludes the proof. \( \square \)
5. Convergence in the regular case

In this section we prove the uniform convergence of the collocation method for linear equation (27) assuming that the spectral radius of the transfer matrix \( M \) is exactly 1, but the eigenvalues on the unit circle have equal algebraic and geometric multiplicities. We call this regular case. Then the operators \( P_N \) are not uniformly bounded if we consider them mapping in some space \( C^K \).

Assume in this section that \( d \geq 0 \). Let us note that, due to the step-by-step character of the interpolation operators \( P_N \), they could be considered acting from \( C^K[0, jh] \) to \( C[0, jh] \) for all \( j = 1, \ldots, N \).

Lemma 13. Let the kernel \( K \) in (27) be bounded on \( S \) and such that \( K \in \mathcal{K}(C, C) \) and \( K \in \mathcal{L}(C, C^1) \). Assume \( r(M) = 1 \) with \( r = 1 \). Then \( I - P_N K \) converges to \( I - K \) stably in \( C \).

Proof. Taking \( k = 1, \ell = 0 \) and regarding \( r = 1 \) in (12) we see that, for any \( f \in C \), \( P_N K f \rightarrow K f \) in \( C \) and this means also \( I - P_N K \rightarrow I - K \). To prove the stability of this convergence, we show that, for sufficiently large values of \( N \), \( ||(I - P_N K)^{-1}||_{C \rightarrow C} \leq \text{const.} \)

Denote \( L = \sup_{(t,s) \in S} |K(t,s)| \). Then, for any \( u \in C \) and \( 0 \leq \tau \leq \sigma \leq T \), we have

\[
\|Ku\|_{C[\tau,\sigma]} = \max_{\tau \leq t \leq \sigma} \left| \int_{\tau}^{\sigma} K(t,s)u(s) \, ds \right| \leq \max_{\tau \leq t \leq \sigma} \left| \int_{\tau}^{\sigma} |K(t,s)| \, ds \right| \|u\|_{C[\tau,\sigma]} \leq L(\sigma - \tau) \|u\|_{C[\tau,\sigma]} \tag{33}
\]

which means that \( \|K\|_{C[\tau,\sigma]} \rightarrow C[\tau,\sigma] \leq L(\sigma - \tau). \)

Take first \( \delta_0 = 1/(4L) \). Using \( \omega(f', h) \leq 2\|f\|_{C^1} \) and (12) we get

\[
\|P_N f - f\|_{C[0,jh]} \leq \frac{1}{4\|K\|_{C \rightarrow C^1}} \|f\|_{C^1[0,jh]} \quad \forall f \in C[0, jh] \tag{34}
\]

for any \( jh \leq \delta_1 \) if \( \delta_1 > 0 \) is sufficiently small independently of \( N \). Let \( \delta = \min(\delta_0, \delta_1) \). Now, for \( jh \leq \delta \), inequalities (33) and (34) yield

\[
\|P_N K\|_{C[0,jh] \rightarrow C[0,jh]} \leq \|P_N - I\|_{C[0,jh] \rightarrow C[0,jh]} \|K\|_{C[0,jh] \rightarrow C[0,jh]} + \|K\|_{C[0,jh] \rightarrow C[0,jh]} \leq 1 + 1 = 2.
\]

hence

\[
\|(I - P_N K)^{-1}\|_{C[0,jh] \rightarrow C[0,jh]} \leq 2. \tag{35}
\]

Take arbitrary \( u \in C \), let \( f = (I - P_N K)u \). If \( jh \leq \delta \) then we get by (35)

\[
\|u\|_{C[0,jh]} \leq 2\|f\|_{C[0,jh]} \leq 2\|f\|_{C}. \tag{36}
\]

Denote \( \delta_h = \max(nh: nh \leq \delta, n \text{ even}) \). Observe that, as \( \delta > 0 \), it holds \( \delta/2 \leq \delta_h \leq \delta \) for all sufficiently small \( h \). Set \( T_\ell = \delta_h/2, \ell = 0, 1, \ldots \). We will show that

\[
\|u\|_{C[0,T_{\ell+1}]} \leq 2(\text{const.}\|u\|_{C[0,T_\ell]} + \|f\|_{C}). \tag{37}
\]

Fix \( \ell \geq 1 \). Take a function \( \varphi \in C^K[0, T] \) such that \( 0 \leq \varphi(t) \leq 1 \) and

\[
\varphi(t) = \begin{cases} 1, & t \leq T_{\ell-1}, \\ 0, & t \geq T_\ell. \end{cases}
\]

Denote by \( M_\varphi \) the multiplication operator acting as \( (M_\varphi u)(t) = \varphi(t)u(t) \). Then we may write

\[
(I - P_N K M_{1-\varphi})u = P_N K M_\varphi u + f.
\]

Introduce the shift operator \( S_\ell \) mapping any function \( g:[0, T_{\ell+1}] \rightarrow \mathbb{R} \) to a function \( S_\ell g:[0, \delta_h] \rightarrow \mathbb{R} \) by \( (S_\ell g)(t) = g(t + T_{\ell-1}), t \in [0, \delta_h] \). Since \( (K M_{1-\varphi} g)(t) = 0 \) for \( t \in [0, T_{\ell-1}] \) and \( g \in C[0, T_\ell] \), taking also into account (33), we have
Remark 15. This gives at once that the operator $I - P_N K : C \to C$ is one-to-one. Note that $P_N : C^1 \to C$ is, in fact, finite dimensional, hence compact and $P_N K : C \to C$ is compact, too. Then by Fredholm alternative, $I - P_N K$ is onto. The estimate (38) gives then uniform boundedness of $(I - P_N K)^{-1}$. The proof is complete. □

Note that, under the assumptions of Lemma 13 about $K$, for any $f \in C$, Eq. (27) has unique solution. From Theorem 4 with Remark 7 and Lemma 13 we immediately obtain

**Theorem 14.** Under the assumptions of Lemma 13 Eq. (28) has unique solution $u_N$ for any sufficiently large $N$. Suppose the solution $y$ of (27) is such that $P_N y \to y$ in $C$ (this is equivalent to $P_N f \to f$ in $C$). Then the sequence $u_N$ converges to $y$ in $C$ with the estimate

$$C_1 \| P_N y - y \| \leq \| u_N - y \| \leq C_2 \| P_N y - y \|.$$

**Remark 15.** It is easy to see that, if $d = -1$ or $d = 0$, the matrices $V$ and $V_0$ (hence, the matrix $M$, too) do not change if we use an arbitrary grid. For these values of $d$, our results are valid in the case of quasi-uniform grid, i.e. whenever $\max_{1 \leq n \leq N} (t_n - t_{n-1}) / \min_{1 \leq n \leq N} (t_n - t_{n-1})$ remains bounded as $N \to \infty$.

As an example of application of Theorem 14 we have the following.

**Corollary 16.** Assume that $K \in C^{m+d+1}(S)$ and $f \in C^{m+d+1}[0, T]$ for some $d \geq 0$ and $m \geq 1$. Let the matrix $M$ in (10) be such that $\tau(M) = 1$ with $r = 1$. Then (27) has a unique solution $y$, for sufficiently large $N$ Eq. (28) has a unique solution $u_N$ and the error estimate

$$\| u_N - y \|_{\infty} \leq \text{const} \, h^{m+d}$$

holds.

**Proof.** The proof is completely analogous to the proof of Corollary 12. □
6. Examples

In order to apply the convergence results of Sections 4 and 5 one needs to analyse the eigenvalues of the transfer matrix $M$ in (10). Several particular cases of $d$ and $m$ in connection with collocation parameters $c_i$ are analyzed in [18]. Therein, the transfer matrix, say $\tilde{M}$, is obtained from another implementation where the collocation equations on neighboring subintervals are subtracted. The characteristic equation $\det(M - \lambda I) = 0$ is equivalent to

$$(-\lambda)^m \det\left(\frac{A - \lambda I_d}{C}\right) = 0$$

(40)

and $\det(\tilde{M} - \lambda I) = 0$ to

$$(1 - \lambda)^m \det\left(\frac{A - \lambda I_d}{C}\right) = 0$$

where $I_d$ is $(d + 1) \times (m + d + 1)$ upper part of $I$. We see that $M$ has $\lambda = 0$ and $\tilde{M}$ has $\lambda = 1$ with multiplicity at least $m$ and all other eigenvalues coincide. Basing on results of [18] we give a brief overview and add some new examples. The reader can find in [18,19] a comprehensive list of numerical results about almost all cases which will be analyzed below. They are in complete accordance with the theoretical results of the present paper.

Case $d = -1, m \geq 1$. Then $M = 0$ and $r(M) = 0$. Under the assumptions of Corollary 12 the numerical method (5) converges with the rate $O(h^m)$ for any choice of collocation parameters $c_1, \ldots, c_m$.

Case $d = 0, m = 1$ (linear splines). The matrix $M$ besides $\lambda = 0$ has eigenvalue $\lambda = 1 - 1/c_1$. We have $r(M) < 1$ for $1/2 < c_1 \leq 1$ and $r(M) = 1$ for $c_1 = 1/2$. Hence the numerical method converges with the rate $O(h^2)$ for $1/2 < c_1 \leq 1$ (Corollary 12), converges with the rate $O(h)$ for $c_1 = 1/2$ (Corollary 16) and is, in general, unstable for $c_1 < 1/2$ (see Theorem 1).

Case $d = 0, m = 2$. For different parameters $c_1$ and $c_2$ besides $\lambda = 0$ of multiplicity 2 there is third eigenvalue $\lambda(c_1, c_2) = 1 - (c_1 + c_2 - 1)/c_1c_2$. We have $r(M) < 1$ if $c_1 + c_2 > 1$ and $r(M) = 1$ if $c_1 + c_2 = 1$. For collocation point of multiplicity 2 corresponding to $c_1$ the matrix $M$ besides $\lambda = 0$ has eigenvalue $\lambda = (1 - 1/c_1)^2$. Thus, $r(M) < 1$ for $1/2 < c_1 \leq 1$ and $r(M) = 1$ for $c_1 = 1/2$. In this case we have the convergence rate $O(h^3)$ if $r(M) < 1$ (see Corollary 12) and $O(h^2)$ if $r(M) = 1$ (see Corollary 16).

Case $d = 0, m \geq 1, c_m = 1$. Then there is only $\lambda = 0$ as eigenvalue (having algebraic multiplicity $m + 1$ and geometric multiplicity $m$). Thus $r(M) = 0$. This is in accordance with the cases $m = 1$ and $m = 2$ indicated above. Under the assumptions of Corollary 12 the method (5) converges with the rate $O(h^{m+1})$.

Case $d = 1, m = 1$ (quadratic splines). If $c_1 = 1$ then the matrix $M$ has eigenvalues $\lambda = -1, \lambda = 0$ and $\lambda = 1$, thus $r(M) = 1$ with $r = 1$ and hence the collocation method converges with the rate $O(h^2)$ (Corollary 16). For $0 < c_1 < 1$ it holds $r(M) > 1$ and the collocation method (5) is unstable even if the kernel of the integral equation is identically zero (see Theorem 1).

Case $d = 1, m = 2$ (Hermite cubic splines). Considering $c_1 < c_2 = 1$ we get $\lambda = 0$ with multiplicity 3 and, in addition, $\lambda = (1 - c_1)/c_1$. This means that, in the case $c_1 = 1/2$ and $c_2 = 1$, we have $r(M) = 1$ with $r = 1$. The domain where $r(M) \leq 1$ is strictly contained in the set \{(c_1, c_2): 0 < c_1 < c_2 \leq 1, c_1 + c_2 \geq 3/2\}. If we assume $c_1$ to determine double collocation point then, for $c_1 = 1, \lambda = 0$ is eigenvalue of multiplicity 4. More detailed analysis shows that, for $\delta = (3 + \sqrt{3})/6$, $\delta < c_1 \leq 1$ yields $r(M) < 1$ and $c_1 = \delta$ gives $r(M) = 1$ with $r = 1$. In this case we have the convergence rate $O(h^4)$ if $r(M) < 1$ (see Corollary 12) and $O(h^3)$ if $r(M) = 1$ (see Corollary 16).

Case $d = 2, m = 1$ (cubic splines). It is well known that $r(M) > 1$ for any $0 < c_1 \leq 1$. Let us add that the collocation with higher order smooth splines is always unstable [12,19].

References