Assertion and denial: A contribution from logical notations

Ahti-Veikko Pietarinen\textsuperscript{a,}\textsuperscript{*}, Francesco Bellucci\textsuperscript{a,b}

\textsuperscript{a} Tallinn University of Technology, Estonia
\textsuperscript{b} Università di Bologna, Italy

\textbf{Abstract}
This paper presents two major aspects of Frege’s and Peirce’s views on assertion and denial: first, their arguments for the notational choices concerning the representation of assertion and denial in Begriffsschrift (BS) and Existential Graphs (EGs), respectively; and second, those properties of BS and EGs which reflect their inventors’ views on assertion and denial. We show that while Frege’s notation has an \textit{ad hoc} sign of assertion and an \textit{ad hoc} sign of negation, Peirce has a sign of assertion which is also a sign of logical conjunction, and a sign of scope which is also a sign of negation.

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0. Introduction

What is an assertion, how does it differ from a proposition? What is a denial, and how does it differ from an assertion? Philosophers of language and logic have been occupied with such questions at least since Frege’s distinction between assertion and the content asserted, a distinction which is expressed in his notation by means of a sign of assertion (the \textit{Urteilsstrich}). Frege has also been credited with being among the firsts to have made the point that negation is not the polar opposite of assertion.

Yet Frege was not the first and not even the most consequential philosopher of language and logic pronouncing upon assertions and denials. Peirce had a number of points to be made on assertion and denial that call upon new and much belated investigation and assessment. Just like Frege, he invented a novel logical notation that expresses quantificational logic, yet one that in significant ways was different both from Frege’s notation and from the notation that has become the received language of first-order logic.

Frege invented the Begriffsschrift (BS) in 1879 [11], Peirce the Existential Graphs (EGs) in 1896 [26–35]. The present paper investigates two major aspects of Frege’s and Peirce’s views on assertion and denial: first, their arguments for the notational choices in BS and EGs concerning the representation of assertion and denial; and second, those properties of BS and EGs which reflect their inventors’ views on assertion.

\textsuperscript{*} Corresponding author.

\textit{E-mail addresses:} ahti-veikko.pietarinen@ttu.ee (A.-V. Pietarinen), bellucci.francesco@gmail.com (F. Bellucci).

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and denial. Frege’s and Peirce’s notations differ from other logical notations as well as from each other, but the sense of this “difference” has to be closely examined. As far as the representation of assertion and denial is concerned, we show that while Frege’s notation employs an ad hoc sign of assertion and an ad hoc sign of negation, Peirce has a sign of assertion which is also a sign of logical conjunction, and a sign of scope which is also a sign of negation. Though we limit our investigation to Frege’s and Peirce’s notational views on assertions and denial, and to their theories of the BS and EGs, respectively, we believe that our lesson is also a contribution to the contemporary and the more general and largely unresolved question of how equivalently expressive notations of a system of logic can differ in ways that are both philosophically significant and relevant to the development of logical and linguistic theories of meaning. Also, the limitation of our discussion to the analysis of assertions in the contexts of the two theories of the BS and EGs by no means implies that there is no interesting work that both Frege and Peirce did on assertions outside of these logical theories. For example, [1, 5, 6, 18, 20] have all discussed Peirce’s responsibility-taking view on assertions without specific attention to the consequences of his theory of EGs in which propositions are expressed on the sheet of assertion.

The paper is divided in two sections, which are devoted to the topics of assertion and denial, respectively. Each section is in its turn divided into two subsections, devoted to Frege (subsections 1.1 and 2.1) and Peirce (subsections 1.2 and 2.2) on assertion and denial.

1. Assertion

1.1. Frege on assertion

Peter Geach [16, p. 449] has devoted considerable attention to the thesis that a thought may have just the same content whether you assent to its truth or not; a proposition may occur in discourse now asserted, now unasserted, and yet be recognizably the same proposition. Geach terms this the ‘Frege Point’. Another formulation of the Frege Point is by David Bell: ‘propositional identity survives changes in assertiveness’ [3, p. 92]. The Frege Point, Geach argued, is something we need in order to understand modus ponens:

(1) (i) ‘If 𝑝, then 𝑞’
(ii) ‘But 𝑝’
(iii) ‘Therefore, 𝑞’

In order to understand an argument of the form (1), we need to assume that it is one and the same proposition, ‘𝑝’, that occurs asserted in (1.ii) but not in (1.i). The reason is that if ‘𝑝’ occurred asserted in the conditional (1.i), then (1.ii), which simply asserts ‘𝑝’, would contain nothing not contained in (1.i), and thus would be redundant. On the other hand, if the proposition ‘𝑝’ that occurs asserted in (1.ii) were not in some sense the same as the proposition ‘𝑝’ that occurs in (1.i), the argument would be vitiated by equivocation. Thus, in order to reconcile these two facts, we need to assume that while ‘𝑝’ is the same proposition in both (1.i) and (1.ii), yet it occurs asserted in (1.ii) but unasserted in (1.i). And thus the point that a proposition or propositional content may occur in logical discourse now asserted, now unasserted, is established. For reasons that will become apparent in the present paper, we propose to re-label Geach’s Frege point as the ‘Geach Point’.

Besides maintaining the Geach Point, namely that one and the same proposition may occur asserted in some contexts and unasserted in others, Frege also maintained that ‘the distinction between asserted and unasserted occurrence [has to] be exhibited notationally’ [8, p. 152]. We call the thesis that the difference between the case in which ‘𝑝’ occurs asserted and the case in which ‘𝑝’ occurs unasserted in some context has to be notationally expressed, whatever form the notational expression of this difference happens to take, the ‘Dudman Point’. Frege thought the Geach Point to entail the Dudman Point, and for that reason endowed
his notation with a sign of assertion, called ‘Urteilsstrich’ (judgment–stroke), in both the Begriffsschrift [11] and the Grundgesetze [12].

Now, in ‘On Peano’s Conceptual Notation and my Own’ Frege observes:

[W]e must deprive the relation sign of the assertoric force with which it has been unintentionally invested. [...] However, we do still sometimes want to assert something, and for this reason I have introduced a special sign with assertoric force, the judgment–stroke. This is a manifestation of my endeavour to have every objective distinction reflected in symbolism [...] Mr. Peano has no such sign: he, on the contrary, uses his relation signs now with and now without assertoric force, and in fact the principal relation sign invariably carries assertoric force. From this it follows that for Mr. Peano it is impossible to write down a sentence which does not occur as part of another sentence without putting it forward as true [15, pp. 247–248].

Frege is not simply making the Dudman Point here. For it is one thing to claim that the distinction between the occurrence of a sentence as asserted and its occurrence as unasserted has to be somewhat exhibited or expressed in the notation, i.e., whatever form the notational expression of this difference happens to take (this is the Dudman Point). It is quite another thing to claim that the expression of the difference between the two situations has to be indicated by an independent or ad hoc sign. An ad hoc sign of assertion is a sign that only signifies assertion, and thus is only employed to signalize when a given sentence is asserted. As Frege observes in the 1906 ‘Introduction to Logic’, ‘we can express a thought without asserting it. But there is no word or sign in language whose function is simply to assert something. This is why, apparently even in logical works, predicating is confused with judging’ [13, p. 185, our emphasis]. What Frege means is that in natural languages there is no ad hoc sign of assertion, and that good logical notations must not follow natural languages but rather employ some ad hoc sign of assertion. We call this latter point, namely that the distinction between asserted and unasserted occurrence has to be exhibited notationally by an ad hoc sign of assertion, the ‘Frege Point’. The Frege Point and the Dudman Point are related to each other as species to genus: to express the difference between assertive and non-assertive occurrences of sentences by an ad hoc sign of assertion is to express this difference somehow, while the reverse is not true, as we shall notice in what follows. But Frege clearly thought not only that the Geach Point entails the Dudman Point, but also that the Dudman Point entails the Frege Point. If one and the same proposition may occur in discourse now asserted, now unasserted, and the difference between the two occurrences has to somehow be notationally expressed, then this difference has to be notationally expressed by an ad hoc sign of assertion. That is, the Geach Point entails the Frege Point, according to Frege.

In the quotation from ‘On Peano’s Conceptual Notation and my Own’, Frege implicitly distinguishes two cases in which a judgeable content, say ‘\(\neg C\)’, can occur unasserted: either (i) ‘\(\neg C\)’ is part of a larger (truth-functional) context that is asserted, or (ii) it is not. Considering that in the Grundgesetze Frege called the representation of a judgment (Urteil) by means of the judgment–stroke a Begriffsschriftsatz, one can put the matter thus: a judgeable content ‘\(\neg C\)’ can occur unasserted either (i) as part of a larger (truth-functional) context that is a Begriffsschriftsatz, or (ii) as, or as part of, something which is not a Begriffsschriftsatz. Case (i) is the paradigmatic case of an expression used not assertively. What preoccupied Frege in the quotation above is case (ii), and he criticizes Peano’s notation on the ground that it is impossible in that notation to write down a sentence which does not occur as part of another sentence without asserting it.

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1 Though each work contains a different interpretation of the sign of assertion, as explained in [8].
2 ‘Die begriffsschriftliche Darstellung eines Urteils mittelst des Zeichens ‘\(\neg\)’ nenne ich Begriffsschriftsatz oder kurz Satz’ [12, §5].
3 Cf. [2, p. 99].

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Case (ii) is the one in which a judgeable content is supposed, named, or taken into consideration in itself without being, or being part of, a Begriffsschriftsätze. Now, in what cases can this happen? Every sentence of the Begriffsschrift is an assertion, because for Frege an inference always proceeds from true premises, not from hypotheses, and thus from assertions, not from unasserted propositions. Both premise and conclusions of an inference must be Begriffsschriftsätze and therefore a Begriffsschrift chain of inference only contains Begriffsschriftsätze. Non-asserted contents of the kind (ii) can only occur outside of the Begriffsschrift, only when sentences of the Begriffsschrift are supposed or mentioned in another language. For example, when Frege in §7 of [11] explains the meaning of the Begriffsschriftsätze

\[
\begin{array}{c}
A \\
\hline
\hline
B
\end{array}
\]

he says that it means

\[
\begin{array}{c}
A \\
\hline
\hline
B
\end{array}
\]

This latter sentence is not a Begriffsschriftsätze, because in it the judgeable content expressed by ' \[
\begin{array}{c}
A \\
\hline
\hline
B
\end{array}
\]

is not used, but mentioned.\(^5\)

In thus recognizing that an unasserted content in the sense of (ii) is not used in the Begriffsschrift but only mentioned in talking of the Begriffsschrift, we reach a point in which we are equally free to choose to symbolize an assertion or to symbolize a supposition.\(^6\) Both alternatives (marking an assertion, or marking a supposition) are open to us. Frege takes the former route (he marks an assertion) but needed not to. Frege’s objection to Peano, that for him ‘it is impossible to write down a sentence which does not occur as part of another sentence without putting it forward as true’ [15, pp. 247-248], is actually an objection to the effect that Peano’s notation does not distinguish the case in which a judgeable content is supposed from the case in which it is asserted. But to remedy this defect Peano has only to carefully enclose the judgeable content within quotation marks when it is supposed. Thus Peano would write

\[
4 'a \supset b'
\]

when he wants to suppose the implication of b by a, while he can use the plain

\[
5 a \supset b
\]

when he wants to assert that a implies b. Likewise, Frege could write

\[
6 'a \supset b'
\]

when he wants to suppose the implication of not-A by B, while he can use the plain

\[
7 a \supset b
\]


\(^5\) On the use/mention distinction see [53, §4].

\(^6\) Cf. [2, p. 85; 56, p. 166].
when he wants to assert that \( B \) implies not-\( A \). In other words, instead of indicating when a content is asserted, Peano and Frege might indicate when it is not asserted, according to the distinction of use and mention of an expression and to a convention to indicate this distinction (e.g., by means of quotation marks). This produces a modified version of the Begriffsschrift, call it BS', in which writing

\[
\begin{array}{c}
B \\
\hline
A
\end{array}
\]

(8)

means the assertion that ‘if \( A \), then \( B \)’, and in which the supposition of ‘\( \overline{A} \)' is indicated, as here, with quotation marks. Thus, as far as our case (ii) is concerned – a judgeable content which occurs non-asserted without being part of a larger truth-functional context that is asserted – Frege’s ‘endeavor to have every objective distinction reflected in symbolism’ can easily be satisfied without introducing an ad hoc sign of assertion, by simply introducing an ad hoc sign of supposition or mention. Under case (ii), the Dudman Point does not entail the Frege Point.

Nor is this entailment valid under case (i). Peano objected to Frege’s use of the sign of assertion on the ground that it was superfluous:

I do not see the utility of these conventions, which have no equivalents in the Formulario. In fact the several positions that a proposition can have in a formula completely determine what is asserted of it. So in our formulas (scritture)

\[
a, \quad a \supset b, \quad a \supset b \supset c
\]

the first says that ‘\( a \) is true’, the second that ‘\( b \) is deduced from \( a \)’, the third ‘if \( b \) is deduced from \( a \), then \( c \) is true’. This latter does not indicate the truth of \( a, b, c \), nor of \( a \supset b \), but only the truth of the indicate relation between these propositions [23, p. 191].

Peano’s point is that the various positions that \( a \) can have in formulas determine whether it is \( a \) that is asserted or whether \( a \) occurs unasserted in some truth-functional context that is asserted. When \( a \) stands alone, it is asserted; when \( \neg a \) stands alone, it is the negation of \( a \) that is asserted; when \( a \supset b \) stands alone, it is \( a \)’s implying \( b \) that is asserted; and so on. We call this the ‘Peano Point’. What the Peano Point amounts to is that in any event when something is asserted it is the main connective of the formula that is asserted. Therefore, to determine whether a sentence \( S \), simple or complex, is asserted in a truth-functional context \( T \), it is sufficient to check whether the main connective of \( S \) coincides with the main connective of \( T \). Instructions to this effect are obviously trivial but support the Peano point: since what is asserted in all events is the main connective of a formula, there is no need for an ad hoc sign of assertion as long as you have a means to indicate the main connective. And since it is simply detrimental for the purposes of logic to express sentences without indicating the scope of operations and therefore the compositional structure of formulas, a sign of assertion becomes, with Peano, completely superfluous.7

Under the case (i), Frege’s argument against Peano may appear as follows. If (8) were the correct Begriffsschriftsatz for (9), we would no longer be able to distinguish the occurrences of ‘\( \overline{A} \)' as non-asserted (as in 11) from its asserted occurrences (as in 9). Nothing distinguishes the occurrences of ‘\( \overline{a} \)' in (8) and (11).

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7 Cf. Wittgenstein: ‘Frege’s assertion sign marks the beginning of the sentence. Thus its function is like that of the full-stop. It distinguishes the whole period from a clause within the period. If I hear someone say ‘it’s raining’ but do not know whether I have heard the beginning and end of the period, so far this sentence does not serve to tell me anything’ [58, §22]. Cf. also Geach: ‘the assertoric force of a sentence is thus shown by its not being enclosed in the context of a longer sentence’ [16, p. 456].
But the Peano Point shows that this argument is fallacious. In fact, it is always the content corresponding to the upper horizontal stroke which is asserted. So the difference between $\bar{\neg}_1$’s being asserted in (8) and its not being asserted in (11) does have a corresponding sign in the notation, that is, $\bar{\neg}_2$’s main connective is identical to the main connective of (8) and not to that of (11). We already have a convention to determine what does and what does not get asserted in a formula without introducing an ad hoc sign for it. We simply exploit a basic and necessary fact of our notations, namely that we always need to signalize what is the main connective of the formula. If one accepts the Peano Point, then even under case (ii) the Dudman Point does not entail the Frege Point. As observed, the Dudman Point and the Frege Point are related to each other as genus to species, so that while the Frege Point implies the Dudman Point, the Dudman Point does not imply the Frege Point.

In reply to the Peano Point, Frege wrote:

I also do not regard the mere counting of primitive signs as an adequate basis for judging how profound the underlying analysis is. I have, e.g., the sign $\mid$, the judgment stroke, which serves to assert something as true. You have no corresponding sign, but you acknowledge the difference between the case where a thought is merely expressed without being put forward as true and the case where it is asserted. Now if the lack of such a sign in your conceptual notation had the effect that, upon close scrutiny, the number of your primitive signs turned out to be smaller, this would not allow us to conclude that your analysis was more profound; for the objective difference remains even if it is not mirrored in the signs [14, p. 117].

This remark is the upshot of an epistolary discussion between Frege and Peano concerning the number of primitive signs of their respective systems, and the ‘profoundness’ of the analysis that each was able to carry. Frege concedes to Peano that everything one can do with the Begriffsschrift with the sign of assertion one can also do with the Begriffsschrift without the sign of assertion (that is, with our BS’). That is, Frege concedes that Peano may be right that the expressive power of the notation may remain untouched by the removal of the judgment stroke. Yet, he argues, there still is an objective difference (the Geach Point) that ought to be ‘mirrored in the signs’ (the Dudman Point).

As it stands, this argument is insufficient to establish the Frege point on the sole basis of the Geach and Dudman Points. An ‘objective difference’ that needs to be ‘mirrored in the signs’ should be a difference between things that can both occur in the Begriffsschrift, and not a difference between how a sentence can occur in the Begriffsschrift (as a Begriffsschriftsatz) and how it can occur outside the Begriffsschrift (as a supposition or mention, our case (ii) above). If this point is admitted, and if some device equivalent to quotation marks is adopted, then the judgment stroke introduces no difference at all in the Begriffsschrift. If Frege still wants any difference internal to the Begriffsschrift to be ‘mirrored in the signs’ – if he wants any objective difference between Begriffsschriftsätze to be so mirrored (our case (i) above) – he is simply mistaken that such a difference is not already mirrored in the signs before the introduction of the convention of the judgment stroke. That is, he is mistaken in holding that the Dudman Point entails the Frege Point. Adding the sign of assertion to a Begriffsschrift formula would be analogous to, and notationally as significant as, say, adding the parentheses to a formula in Polish notation:

(12) $\text{Ko}(Kbc)$
(13) $\text{K}(Kab)c$
The assertoric force of a sentence is already shown by its not being enclosed in the context of a longer sentence, and if you already have a device to show how the longer sentence is structured, you already know what is asserted. The signs of the Begriffsschrift can distinguish asserted from unasserted content without introducing the judgment stroke: each of the horizontals appearing in a Begriffsschrift formula other than the upper horizontal lacks assertive force; the only horizontal that has assertive force is the upper one. In notations that lack any ad hoc sign of assertion, the marking of the compositional structure is the marking of assertion (the Peano Point).

The Peano Point shows that, even when a notation lacks an ad hoc sign of assertion, that notation must have, on pain of inconsistency, an instrument that makes the ‘objective difference’ between asserted and unasserted contents ‘mirrored in the signs’. Both the Dudman and the Peano Points presuppose the Geach Point. And Frege was certainly right to hold the Geach Point. He was also right that the Geach Point should entail the Dudman Point. But he was wrong that the Dudman Point entails the Frege Point. The Peano Point shows that the Dudman Point does not entail the Frege Point.

1.2. Peirce on assertion

We are now ready to let our second principal actor to this story to speak up. The following passage from ‘Κανά στοιχεία’, a text composed in 1904 but published posthumously only in 1976, unmistakably shows that Peirce endorsed the Geach Point:

A proposition [...] is not to be understood as the lingual expression of a judgment. It is, on the contrary, that sign of which the judgment is one replica and the lingual expression another. But a judgment is distinctly more than the mere mental replica of a proposition. It not merely expresses the proposition, but it goes farther and accepts it. I grant that the normal use of a proposition is to affirm it; and its chief logical properties relate to what would result in reference to its affirmation. It is, therefore, convenient in logic to express propositions in most cases in the indicative mood. But the proposition in the sentence, “Socrates est sapientem,” strictly expressed, is “Socratem sapientem esse.” The defense of this position is that in this way we distinguish between a proposition and the assertion of it; and without such distinction it is impossible to get a distinct notion of the nature of the proposition. One and the same proposition may be affirmed, denied, judged, doubted, inwardly inquired into, put as a question, wished, asked for, effectively commanded, taught, or merely expressed, and does not thereby become a different proposition [36, p. 248].

In Peirce’s logical algebras, writing a formula signifies the assertion that the content corresponding to that formula is true in the universe of discourse. Thus, ‘p’ is asserted in (14), while in (15) it is the conditional ‘if p, then q’ that is asserted. Neither ‘p’ nor ‘q’ is asserted in (15), only their conditional relation is.

\[
(14) \quad p \\
(15) \quad p \iff q
\]

It is clear that while Peirce endorses the Dudman Point, namely that the difference between the case in which a formula occurs asserted and the case in which it occurs unasserted has to be notationally expressed, yet he does not endorse the Frege Point, namely that the difference between the case in which a formula occurs asserted and the case in which it occurs unasserted has to be notationally expressed by an ad hoc sign of assertion. Peirce does endorse the Peano Point, namely that the various positions that a formula can take are

\[8\] The Geach Point entails the Dudman Point at least in a normative sense, so that the latter may be regarded as a normative requirement for what a good notation is. We owe this suggestion to an anonymous reviewer of this journal.
have determine whether it is asserted or whether it occurs unasserted in some truth-functional context that is asserted. For this reason, no *ad hoc* sign is employed in Peirce’s algebras to indicate that ‘\( p \)' is asserted in (14) and not in (15).

This feature is brought to prominence in one of the systems of logical graphs invented by Peirce, the Existential Graphs (EGs).\(^9\) In EGs, the scrib ing of a graph on a blank sheet, called the Sheet of Assertion,\(^10\) corresponds to the assertion of the propositional content expressed by the graph. Thus, if I write ‘It rains’ on the sheet (16), I assert that it rains. If I write both ‘It rains’ and ‘It thunders’ (17), I assert that it both rains and thunders. Juxtaposition on the sheet expresses logical conjunction.

\[(16) \quad \text{It rains} \]
\[(17) \quad \text{It rains} \quad \text{It thunders} \]

In Peirce’s notations, both algebraical and graphical, the writing of ‘\( p \)' signifies the assertion that ‘\( p \)' is true. This is the same thing as to say that the sheet upon which the logician writes her formulas is a sheet of *assertion*, that is, all and every well-formed formulas written upon it are thereby asserted. By contrast, Frege’s ‘sheet’, the pseudo-sheet upon which he writes his Begriffsschrift formulas, is not a sheet of assertion. The mere writing down of a Begriffsschrift formula is not yet to assert it.

Peirce has an argument for the convention that a sheet is always to be considered as a sheet of assertion. The argument is contained in an unpublished manuscript dating circa 1906 [39]. The argument is intended to prove that the sheet is the necessary sign of assertion in any conceivable notation, and further that any such sheet, conceived as a sign of assertion, is also necessarily a sign of logical conjunction.

If one undertakes to diagrammatize reasoning, one can manifestly hardly do otherwise than make some surface what I have called the Phemic Sheet, and endow it with the property that whatever proposition is *expressed* upon it shall thereby be understood to be *asserted*, and that whatever proposition is to be asserted shall be scribed on that Sheet. Moreover, all reasoning being dialogic, we are entitled to talk of a Graphist and an Interpreter. But Graphist and Interpreter must from the outset of their discussion already thoroughly understand each other to accept a number of truths as unquestionable. Otherwise, they could have no more discourse together than an inhabitant of the planet Mercury and a Neptunian. Their very acceptance of the Phemic Sheet acknowledges some of these axioms; and thereby that Sheet becomes the sign and expression of those truths. One may say that the blank of the Sheet is the Graph of those unquestioned assumptions. Accordingly, the very first proposition that is scribed on the Sheet is scribed on a sheet already bearing a proposition; and that, no matter what the new proposition. Thus, consistency (and of course a good system of diagrammatization of anything, and above all of reasoning, must be consistent) requires the general acceptance of the principle that every proposition scribed upon the Sheet is asserted independently of every other that may already be scribed there, unless the new proposition is represented in its expression as in some way dependent upon the old ones. This is the principle of copulation, that AB on the Phemic Sheet asserts both A and B independently of each other [39, pp. 1–3].

We can divide Peirce’s argument in this passage into three steps. In the first step, he argues that since it is usually implicitly assumed that when something that can be used to make an assertion is written down on a piece of paper it is in fact understood as making the assertion, then one might as well adopt such implicit

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\(^9\) Original expositions of EGs are in [26, 27, 29, 33–35, 40, 46]. On later studies on EGs see [4, 48–52, 54]. EGs developed out of another system of logical graphs named ‘Entitative Graphs’, which Peirce first published in [28].

\(^10\) The Sheet of Assertion was re-named, after 1906, the ‘Phemic Sheet’. Cf. [39, 41, 42, 45].
assumption as an explicit convention of the system of notation. If as I wake up in the morning I notice a little piece of paper on the kitchen table saying ‘The plants have been watered’ in my spouse’s handwriting, I assume that it is thereby asserted that the plants have been watered, not that it is thereby asked whether the plants have been watered, nor that it is thereby commanded that the plants be watered, nor that the sentence has been written down as an exercise in handwriting. This happens, Peirce claims, because writing ‘p’ is usually intended as the assertion that ‘p’ is true. Thus, and this is Peirce’s first argumentative step, one might as well exploit this implicit assumption and turn it into an explicit convention of the notation: writing something on the sheet means asserting it. In point of fact, all notations, with the exception of those which employ an ad hoc sign of assertion, do adopt this convention.

Now, if the sheet is considered as a sign of assertion, then it must also, in itself, signify an assertion. In conformity with his dialogical and game-theoretical interpretation of logic, Peirce uses the imaginary figures of the Graphist and Interpreter of the graphs to make the point: the Graphist is the utterer who asserts propositions that are scribed on the sheet; the interpreter either accepts or rejects the modifications put forward by the Graphist. Now Graphist and Interpreter must from the outset already understand each other to accept a number of truths. One of these truths is, of course, that they are talking of one and the same universe of discourse. This means that if the Graphist scribes the sentence ‘The plants have been watered’ on the sheet, the Interpreter has to understand that the Graphist has asserted, of their common universe of discourse, that it is true of that universe that in it the plants have been watered. Likewise, in reading my spouse’s note I must assume that it refers to our plants in the garden, not to the plants that Caterina de’ Medici commanded to her servants to water in June 1588. All the assumptions, which Graphist and Interpreter must share in order for them to be able to engage in reasoning, are symbolized by the sheet. The sheet represents ‘all the truth that was well-understood between Graphist and Interpreter to be taken for granted’ [37]. But then, being the sum total of the assumptions with which we begin any logical theorizing, the sheet must also, in itself, be of the nature of an assertion: ‘The blank sheet must have the force of a proposition asserting whatever is taken for granted in every discussion and does not need to be explicitly set down’ [40, p. 894]. The sheet is at the same time a predicate (what Peirce calls a ‘rHEMA’ in 1903, and a ‘seme’ in 1906) and a proposition (what Peirce calls a ‘dicisign’ in 1903, and a ‘pheme’ in 1906): the sheet ‘is to be, at once, a seme of The Truth, in the sense of the widest Universe of Reality, and a pheme of all that is virtually taken for granted between the Graphist and the Interpreter from the outset of the discussion’ [41, p. 41]. The first of the basic rules in Peirce’s graphical system indeed is that we always have a logical right to a blank sheet. This rule is also expressed as an axiom of the system of transformations: we are always entitled to assert the empty form of a proposition, such as ‘What I aver is true’ [27]. Having a logical right to a blank sheet means that the proofs begin with the SA, that is, an assertion of tautology (T, verum).

By this second argumentative step Peirce has thus drawn the consequence that the sheet itself is an assertion, namely the assertion that the Graphist and the Interpreter of the graphs are discussing about one and the same universe of discourse. He is now in the position to draw a further conclusion from the first two argumentative steps. If scribing a proposition on the sheet counts as its assertion, and if the sheet is already, in itself, an assertion, then ‘the very first proposition that is scribed on the Sheet is scribed on a sheet already bearing a proposition’. That is, scribing ‘p’ on the blank sheet is equivalent to adding ‘p’ to a proposition already scribed, namely to the proposition that there exist a universe of discourse of which ‘p’ is asserted. But then – and this is the conclusion of the third argumentative step – every proposition scribed upon the sheet is asserted independently of every other that may already be scribed there. If the very first proposition that is scribed on the sheet is scribed on a sheet already bearing a proposition, then this must be true of any subsequent proposition that is scribed thereupon. The first proposition scribed – the first assertion made about the universe – is thus no different from any other proposition subsequently scribed (from any other
assertion subsequently made) upon the sheet. But since the first proposition is asserted independently of any other proposition besides the assertion of the existence of a certain universe of discourse, so must any subsequent proposition. We have seen that since the sheet is already a proposition, the first proposition scried is no different from any subsequent proposition. Thus, if the first is asserted independently of any other proposition, so must any other proposition thereupon scried be asserted independently of the others.

All propositions scried on the sheet are asserted independently of all the others. But to say this is to say that the sheet is, at once, a sign of assertion and a sign of logical conjunction. To elucidate this point we should briefly return to Frege. According to Bell, the test for deciding whether a propositional content occurs assertively within a given context is the test of exportation: ‘if the proposition can be removed from that context without further ado, and displayed on a line in a proof by itself, then that proposition is asserted in the original context’ [3, p. 104]. According to this test, ‘p’ occurs asserted in ‘p and q’, because it can be exported without further ado: the truth of ‘p’ entails the truth of ‘p’.

Now, what is noteworthy is that it is the conventions of Peirce’s graphs that straightforwardly embody the rule of conjunction elimination. To adopt the convention that the sheet is a sign of assertion is to allow that every proposition scried is in the same relation to the sheet as every other, and thus is asserted independently of every other that may already be scried there. This is nothing less and nothing more than the rule of conjunction elimination: whatever entire graph is at any time scried on the sheet, each of its juxtaposed subgraphs possesses its meaning (truth-value) independently of any other juxtaposed subgraph that is scried thereupon. Thus the sheet, in embodying the rule of conjunction elimination, must signify conjunction, because a syntactic feature of a notation cannot embody a rule for a truth-functional operation without thereby representing that truth-functional operation. On the other hand, the reason why ‘p’ occurs unasserted in the EGs representation of ‘p or q’,

\[
\begin{array}{cc}
p & q \\
\end{array}
\]

is that since they are isolated from each other by cuts around them, the disjuncts are not continuously connected with one another and hence not independently asserted on the sheet.

Dummett has argued, contra Bell, that the Fregean judgment–stroke ‘cannot be taken to attach to any subsentence, even when the asserted complex sentence would logically justify the assertion of some subsentence’ [10, p. 493]. Dummett’s argument is that the entailment of ‘(Δ P) & (Δ Q)’ by ‘Δ (P & Q)’ should not mislead us into thinking that the sign of assertion can occur within the scope of a truth-functional operator. If conjunction were the only truth-functional operator in the language, we could always take it as operating on sentences having the sign of assertion attached, for an assertion of a conjunction is plainly equivalent to a conjunction of assertions [9, p. 336]. But an assertion of a disjunction is not equivalent to a disjunction of assertions, and an assertion of a conditional relation between two propositions is not equivalent to a conditional relation between two assertions. It cannot be plausibly maintained that ‘Δ (P & Q)’ is equivalent to ‘(Δ P) ∨ (Δ Q)’, or that ‘Δ (P ⊃ Q)’ is equivalent to ‘(Δ P) ⊃ (Δ Q)’. Thus, for consistency’s sake the sign of assertion cannot be taken to occur within the scope of a truth-functional operator.

It is evident that Dummett’s argument against Bell only holds if it is referred to a language in which the sign of assertion is an ad hoc sign of assertion, namely a sign that only signifies assertive force. That is to say, Dummett’s argument is based on the implicit assumption that the Dudman Point entails the Frege Point. If the sign of assertion is intended, following the Frege Point, as an ad hoc sign of assertion, then Dummett would be right were he to maintain that such a sign always has to be attached to sentences as wholes or to entire graphs and never to their proper parts, even when this would be justified by rules of inference like conjunction elimination. But if the sign of assertion is taken in the way Peirce proposed, that
is, as both a sign of assertion and a sign of logical conjunction, then this sign readily embodies the rule of conjunction elimination, and Bell’s exportation test is perfectly in order.

One may object that this is to confuse the illocutionary with the locutionary level of analysis, for while assertion concerns the former, logical conjunction only concerns the latter. Thus, it may be objected, having one and the same sign for both is a symptom of ignoring the Geach Point. However, to accept Bell’s exportation test is not to accept that the sign of assertion must always occur in the scope of conjunction, but only that it can. Conjunctions may have illocutionary force, but not by necessity. Peirce’s logical graphs are a case in point.

In a draft of the projected and never completed Minute Logic Peirce proposes another argument for the sheet of assertion in his graphs.

[T]he algebra of the copula makes use of no general algebraic sign except \(-\) or no other ‘particularly noticeable.’ It does, however, employ two signs without which no algebra would be possible, and which are the most efficient tools of every algebra, although the treatises on algebra find no need of dwelling upon them. The first is the operation of writing down an expression on a particular sheet, or under some well-understood circumstances, to show that you mean to assert its truth, at least, conditionally; and further, that, having written such expressions, you mean that any one or more of them would remain true, though the others were erased, – I do not say, denied, but simply erased. Without this operation, algebra would obviously be impossible. The second universal sign consists in enclosing a sign of operation with its subjects in parentheses, or in otherwise indicating that it, or its result, is to be taken as a subject of a further operation. Doing this and the like of this might almost be said to be the stuff of which mathematics is composed [...] An algebra without a sign of this import would be mathematically impotent [32, pp. 57–58].

Peirce discovered the functional completeness of the joint denial for Boolean algebra in 1880 [24], rediscovered by Sheffer in 1913 [55]. The possibility of defining all operations of the propositional calculus by means of the joint denial shows, according to Peirce, that none of those operations is necessary in logical algebra. A similar view was held by Wittgenstein, who wrote in the Tractatus that the possibility of a crosswise definition of the primitive signs shows that these are not primitive signs [57, 5.42]. But for Peirce, two fundamental signs must be present in any conceivable notation for logic. The first sign which is necessary in any conceivable notation for logic is the sheet of assertion. The reason is that in any conceivable logical notation you need to write down what you assert; a notation in which you need not write down what you assert is inconceivable. We can take the sheet of assertion to play the role of what in topology is meant by the ambient space. The import of the two is the same: that there is nothing (propositions in logic or abstract forms in topology) that could be expressed without scribing them in something, in some media, manifold, space or sphere of mathematical imagination or cognition. This is the first necessary sign of any conceivable notation. In the second Lowell Lecture of 1903 Peirce explains that it would be physically impossible that a blank should not accompany every graph. The truth is that the system of existential graphs was intentionally contrived so that this matter should take care of itself [33]. Since in any conceivable logical notation you need to write what you assert, then we might as well adopt the convention that writing a formula means asserting it, or, in an alternative but equivalent formulation, the convention that the sheet upon which we write or scribe formulas and graphs is a sheet of assertion. Peirce explains that the graphs had been invented with the purpose that ‘this matter should take care of itself’, that is, with the purpose of having the first necessary sign (writing or scribing something) also signify assertion (writing or scribning something means asserting it). The sheet in the logical graphs ‘takes care of this matter’ because it exploits the fact that whatever you want to express, you have to write it down, and thus there is no reason not to adopt this necessary sign also as a sign of assertion. Given the three-step argument presented in [39], this also implies that a sign of assertion so designed also has to function as the sign of logical conjunction.

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Fregé’s notation does not ‘take care of this matter’: since he did not consider the medium on which he wrote his formulas as a sheet of assertion, he needs a sign of assertion. And thus he is forced to introduce some ad hoc sign of assertion (the Urteilsstrich) only because he deprives the sheet of its natural assertive nature, which then needs to be restored by a sign of assertion.

We can make the case for Peirce even stronger by considering quantified contexts. In the standard theory of Beta graphs, which is Peirce’s system for first-order predicate logic, existential quantification is represented by continuous lines connecting predicate terms. Lines represent quantification, predication, and identity with one and the same sign. Lines can also branch and have loose ends. In its most basic outfit, the line is a one-dimensional dot. A heavy dot that is marked on the sheet stands for an individual object that exists in the universe of discourse. When a dot moves, its trail shows the identity as a line or a connection. A continuous connection between the line’s extremities denotes the identity of those individuals for which the extremities stand. Now a dot, when scribbled on the sheet of assertion, means that something exists in the universe of discourse which is identical to itself. The dots represent self-identity, namely that the individuals they stand for are well defined. Two dots on the sheet may or may not be identical to each other, but they are identical to themselves each. That individuals are well defined is a precondition for something that exists to be assertible. For when a dot is scribbled on the sheet, this very act of scribbling involves ‘assuming the responsibility of asserting that the object denoted by that sign is in the universe of existents’ [40]. Peirce means that ill-defined individuals, not representable by any dot, are not scriptible on the sheet and thus not assertible.

All this is known from what Peirce has stated in those of his papers that have been published so far. The act of scribbling is to make an assertion of an individual existence. That is, the one who scribbles the dot has to bear the responsibility for an individual existence of that something in the universe of discourse. But there are nevertheless important further details to Peirce’s ideas about assertions in the Beta system and to his notation for quantifiers which have not been noticed before. In ‘The Principles of Logical Graphics’, one of the early and still unpublished papers on logical graphs (1896–1899), Peirce experiments with a version of the notation in which the ends of the lines have a little swelling or a loop. This is his Principle 5:

**Principle 5.** A heavy line terminating abruptly without any head or finish shows a complete assertion is not intended. But a swelling, or button at the end is an indefinite index [31].

\[
\text{(19) } \bigcirc \text{sings} \quad \text{(20) } \bigcirc \text{sings (in Peirce’s hand)} \quad \text{(21) } \bigcirc \text{sings}
\]

Of these three examples given by Peirce, the graphs in (20) and (21) make assertions (‘In the universe of persons, something sings’) but the graph in (19) does not. Since the graph in (19) lacks the dot at the free end of the line, the individuality it represents is not well-defined, and thus the graph does not assert anything. Graphs in (20) and (21) are assertions, while (19) is what Peirce called a ‘rhema’: attaching a standard line to it does not yet make an assertion out of it. Only a line that has such a swelling or a loop can be used to make an assertion. How can this be so?

As usual, also a notational detail that might at first seem of minor relevance has an important logical reason. As the sign of well-defined individuality, even more primary than the dot is the loop.

\[
\text{(22) } \bigcirc \text{ (in Peirce’s hand)}
\]

Although the loop as such is a rare feature of Peirce’s standard presentation of his EGs, yet it is the sign that at bottom signifies self-identity and thus well-definedness of an individual. Now it can happen that the

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12 Cf. [7,54].
size of the loop is reduced until what it surrounds becomes invisibly small, and until what we are virtually left with is a one-dimensional dot. It is this resulting dot that now serves as the common sign of self-identical individuality. (As a matter of practical benefit, a dot surely can while a loop with no outpouchings in it cannot be connected to the hooks surrounding the predicate terms or ‘spots’.) Thus those lines of identities that assert identities between individual existences, as drawn by dots that move about on the sheet, must have those dots at their extremities. The lines deprived of dots are not genuine signs of identity and thus are not really scribed on the sheet according to the basic conventions of the system. Consequently, they cannot be used to make an assertion, either. For no one in her right mind can pretend to bear a responsibility of claiming the existence of individuals that are ill-defined. Only the lines that are constituted by a continuum of dots assert individual existences and identities. So to be notationally precise, such lines always begin and end with dots or loops. Peirce’s own graphs in his manuscripts occasionally show those dots appearing as little bullets at the ends of the lines as they do in (21), but mostly they do not. But even when they do not, they must be assumed to be present.

The real import of the sheet of assertion is thus not that it is a two-dimensional sheet. The real import is that the sheet makes the act of scribing that adheres to the conventions of the system an assertion. This fact can now be confirmed also for the quantification theory of Beta graphs. The act of scribing is to make an assertion, but in doing so it does not introduce any ad hoc sign of assertion, as the act of scribing uses signs that already bear the requisite logical content (such as juxtaposition for conjunction, and dots and lines for existence, identity and predication). The swelling or a dot at the extremities of a line of identity is not an ad hoc sign of assertion, just as the sheet is not, because the dot readily asserts the existence of well-defined individuals in the universe of discourse that the sheet of assertion represents. The dot does not only signify an assertion; it becomes an assertion of the existence of something just when it also functions as a sign of a quantifier.

Hence the Bell exportation test is perfectly in order also for the quantificational cases as presented by Peirce in [31]. Existential instantiation is in graphical terms a predication where a well-defined individual or its name can be substituted for a dot or the end of the line connected to the hook of the spot (predicate term). Scribed on the sheet of assertion, the graph expressing predication is an assertion. Since it stands on the line of proof itself as an assertion, the original graph, namely the spot with a dot or a line connected to its hook, is also an assertion.

It is worth adding that there can be, and indeed Peirce also did consider there to be, special signs that can indeed be used to make assertions about other signs, namely signs that serve as meta-assertional devices. This notation exploits the idea of using graphs to talk about the elements and properties of graphs in the very language of graphs. This discovery matured to him while preparing the Syllabus for his Lowell Lectures [35]. But already five years earlier, he had proposed that meta-assertions be connected to the ovals from the outside:

When we have occasion to write down a proposition not to assert it, but to say something about it, we will draw an oval round it, to show that it is not asserted, thus,

\[\text{You are a good girl}\] \text{is important if true}\]

But the principal thing to be said about a proposition is that it is false. There is no occasion to write that a proposition is true, because it is simpler to assert the substance of the proposition. But we so often have occasions to say

\[\text{You are a good girl}\] \text{is false}\]

and the like, and, so seldom have occasion to say anything else about a proposition, that we may write
You are a good girl

and omit the “is false” [30].

Thus even the meta-assertional devices make do without any ad hoc signs for assertion. The quotation above is from a draft of the Cambridge Conference Lectures. In the delivered version of the same paper, the meta-assertion is notated such that the proposition which is not asserted (encircled in an oval) is connected to the meta-assertional statement of ‘is much to be wished’ with a line.

(23) You are a good girl—is much to be wished

But even that line does not count as an ad hoc sign for meta-assertions. The line merely accentuates the fact that there is the continuity of the sheet which can also be represented by the proximity of the two propositions: the quotational one enclosed within the oval and the ascription to it of the property of being much to be wished. The important thing is that in both examples the office of the oval is as a quotational sign.

Interestingly, then, Peirce’s 1898 notation serves as the bridge that connects the first universal sign, the act of making an assertion carried out as the act of scribing graphs on the sheet, to the second universal sign, the collectional sign that denotes the scope of logical operators.

Peirce arrives at this proposal to connect the act of making assertions as acts of scribing to the logical notion of scope when he was contemplating on the nature of assertions from the point of view of the problem of there being some properties that one cannot assert or attribute to something or to someone, although one could well represent those assertions that do so. He was in turn led to these considerations on the existence of such a property of assertions by his thoughts on ‘widening the universe of discourse’ and the ensuing experiments on new kinds of graphs in which special spots or rhemas are added to the language of graphs that signify meanings such as ‘____ asserts the truth of_____’ or ‘the proposition G is true’ [26]. These meta-graphical elements would later resurface in the theory of Gamma graphs of his 1903 Lowell Lectures, in terms of the higher-order logic of potentials and in his idea of graphs of graphs.

As an example, the Gamma graph in (25) is the assertion that the Alpha graph in (24), which latter represents the conditional de inesse ‘If it hails, it is cold’, is scribed on the sheet of assertion.

(24) it hails
    it is cold

(25) (in Peirce’s hand)

Here ← means ‘is the sheet’, W→Z means ‘Z is the area of the cut W’, Y→X means ‘X is placed on Y’, and U←V (in Peirce’s hand) means ‘U is a graph precisely expressing V’. While Peirce does not proceed to develop much further the logic of assertions fledgling in these proposals, the many examples and the elaborate definitions of the elements of the graphs to analyze what those properties of graphs are that can be scribed and thus asserted testify that he was aware of the significance of the possibility of using his graphical logic towards realizing that goal.

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We leave the precise analysis and a further development of the above suggestions for a future occasion. The main upshot for the purposes of the present paper is that the sheet of assertion is one of the two universal signs without which no algebra would be possible. The other universal sign is what in other contexts Peirce calls ‘collectional signs’. These are signs whose task in the notation is to indicate the scope of operations, and thus the compositional structure of formulas. Collectional signs, in Peirce’s logical graphs, also represent negation.

2. Negation

2.1. Frege on negation

Frege did not take assertion and denial as two opposite or polar types of illocutionary acts. In his logic there is only one illocutionary act which is relevant, and this is assertion. The denial of a proposition corresponds to the assertion of the negation of that proposition. In Austinian terminology, negation is part of the locutionary content of the speech act, not of its illocutionary force.

Frege’s argument for this view is expounded in his 1919 essay ‘Die Verneinung’.\(^\text{13}\) Suppose that negation is the polar opposite of assertion. This would involve seeing negation as part of the illocutionary force and not as part of the thought or content of the speech act. Still, negation must also, at least in some cases, be part of the thought and not of the illocutionary force. For example, in the sentence ‘If the accused was not in Berlin at the time of the murder, he did not commit the murder’, we cannot understand the antecedent as carrying negative assertoric force. In a conditional neither antecedent nor consequent are asserted, only their conditional relation is. And since the negation of a conditional is plainly something different from a conditional whose antecedent is negative, the negation in the antecedent cannot be taken as carrying assertoric force. As Dummett puts it, ‘while it would be possible to interpret those sentences in which the sign of negation was the principal operator, it would be impossible to interpret the negation sign in this way whenever it occurred in a sentence otherwise than as the main operator’ [9, p. 316]. Thus, negation can occur as part of the thought and not as part of the illocutionary force.

Are we to think, then, that there are two distinct kinds of negation, one that occurs at the illocutionary level and one that occurs at the locutionary level? That the answer to this question is negative is shown again by appeal to modus ponens. Let us assume, for discussion’s sake, that there in fact are two kinds of negation, one of which occurs at the illocutionary level, and which may be expressed by the phrase ‘It is false that...’, and one that occurs at the locutionary level, and which is expressed by standard negative particles such as ‘not’ and ‘non’. Now take the following argument:

If the accused was not in Berlin at the time of the murder, he did not commit the murder;
It is false that the accused was in Berlin at the time of the murder;
Therefore, he did not commit the murder.

In this argument, the thought expressed by the antecedent of the first premise and that expressed in the second premise is not the same: for in the antecedent of the first premise the thought is that the accused was not in Berlin at the time of the murder (negation occurs at the locutionary level), while in the second premise the thought is that the accused was in Berlin at the time of the murder (negation occurs at the illocutionary level). But since the thoughts are not the same, the validity of modus ponens is compromised. If we still want the inference to be a valid instantiation of modus ponens, then we have to acknowledge that the second premise contains the same thought as the antecedent of the first, and thus we have to acknowledge that ‘It is false that the accused was in Berlin’ is equivalent to ‘The accused was not in Berlin’, that is, we

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\(^\text{13}\) A very brief sketch of Frege’s 1919 argument is given in [22, pp. 139–140]. See also [9, pp. 316–318].
have to acknowledge that there are not in fact two different kinds of negation [15, p. 384]. Negation must occur as part of the thought and not as part of the illocutionary force, and the illocutionary denial that ‘p’ must be considered equivalent to the assertion of the locutionary negation of ‘p’.

But if the illocutionary denial that ‘p’ is equivalent to the assertion of the locutionary negation of ‘p’, then the concept of ‘illocutionary denial’ is superfluous. Frege explicitly condemns such an approach on the grounds that it contravenes the principle of economy of the language. If we assume two kinds of negation, then we would need (i) a sign of assertion, (ii) a sign of illocutionary denial, and (iii) a sign of locutionary negation. But since (ii) can be expressed by means of (i) and (iii), a more economic notation is one in which (ii) is absent. This line of reasoning is that by which we would be brought to conclude that since existentially quantified assertions can always be expressed as universally quantified sentences, then the concept of existential quantification and its symbols are superfluous. And it was Frege’s idea that, if it is possible to economize, then it is a commandment of perspicuity to do so, as a more economic notation is one that pushes logical analysis further [15, p. 385].

For Frege the sign of negation is only a sign of negation. In the *Begriffsschrift*, the sign of negation is appended to any content that is negated, and its only function in the notation is to negate all contents that are at its right on the horizontal. It is the position that the negation sign has in the conditional structure that determines what content is negated by it. The *Begriffsschrift* has, just like the majority of logical notations, an ad hoc sign of negation. By contrast, in Peirce’s logical graphs the sign of negation is not only, and not even primarily, a sign of negation.

2.2. Peirce on expressed

Negation is expressed in Peirce’s logical graphs by means of ovals that encircle the graphs to be negated. Peirce sometimes calls the ovals ‘cuts’ to signalize that they cut away a portion of sheet from the sheet itself. In Existential Graphs, (26) represents ‘p & ¬q’, (27) represents ‘p & (q & r)’, and (28) represents ‘¬(p & ¬(q & r))’.

\[ (26) \quad p \quad \text{oval} \quad q \]
\[ (27) \quad p \quad q \quad \text{oval} \quad r \]
\[ (28) \quad p \quad q \quad \text{oval} \quad r \]

But the ovals are not simply, and not even primarily, signs of negation. In the *Minute Logic* of 1902 Peirce explains that the ovals indeed combine a number of offices: ‘even when there are no lines of identity, they fulfill three distinct offices, and [...] in introducing these lines we have imposed upon them two more’. They ‘fulfill all five with success’ [32, p. 53]. The first function is a truth-function: the ovals signify negation. Second, they serve as collectional signs or ‘signs of scope’, and thus have a collectional function. Third, they can express all modes of logical combination when position on the sheet is taken to represent assertion or, equivalently, juxtaposition is taken to represent logical conjunction. In the Beta part, the ovals add two further offices: the fourth office ‘is to indicate the order of succession of the identifications’, that is relations of dependence among quantifiers, while the fifth office is to cut continuous segments out of the lines of identity in order to represent non-identities between the individuals denoted by extremities of the line [32, pp. 53–63].

The third office – representing all modes of logical combinations – is a consequence of the first two; the fourth office – representing order of selection – depends on the second office (representing scope); the fifth office – representing non-identity of lines – depends on the first (representing negation). The third, fourth, and fifth offices are derived, the first and second primary. As far as these primary offices are concerned, we can say that in the logical graphs one single sign – the oval – has both truth-functional

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\[ p \quad q \quad \text{oval} \quad r \]

14 See [52] for more details.
and collectional meaning. Peirce is keenly aware of this double role: ‘while the syntax of existential graphs thus needs both a sign of negation and an endless series of collectional signs, there is no reason why a single sign should not perfectly fulfill both these purposes’ [46, p. 908].

The second universal type of sign without which no algebra would be possible is what Peirce calls ‘collectional signs’.\(^{15}\) A notation can exist without a sign for, say, logical disjunction or arithmetical division, but no notation and no algebra can exist without a sign of scope. In Peirce’s logical graphs, the sign of scope is the ‘oval’, which also serves as a sign of negation. Now, this convention, namely that the oval should represent both negation and scope (and in quantificational logic, the order of selection), is a consequence of the convention concerning the sheet of assertion, namely that the sheet represents both assertion and scope. For once the primary convention of logical graphs is adopted that the sheet represents logical conjunction [26], no scope indicators are needed until negation is added to the function of the operators. Trivially, while ‘\(p \& (q \& r)\)’ is equivalent to ‘\((p \& q) \& r\)’, ‘\(p \& (\neg q \& r)\)’ is not equivalent to ‘\(p \& (\neg q \& r)\)’. Peirce was no originator of the idea of having a sign of scope which also signifies negation: the vinculum as sign of scope was known to Descartes and Leibniz, and the vinculum as the sign of both negation and scope was in use in Boolean algebras of logic. Peirce’s real notational invention was that the sheet of assertion serves both as the assertional sign and as the sign of logical conjunction. Topologically, a vinculum in the open-compact space of EGs can only be an oval. The representation of negation can be assimilated with the representation of scope (given the endoporeutic interpretation of graphs) because negation is the only other necessary operator besides the symmetric operator of logical conjunction. While conjunction is commutative and associative and thus takes the symmetric, topologically open-compact sheet as its basic sign, negation introduces orientation and anti-symmetry into the system and is thus coupled or notationally entangled with the sign of scope.

Peirce’s idea was that, of its primary functions, the oval signifies scope primarily and negation secondarily. The order of presentation of the oval’s offices in the Minute Logic would thus require a slight emendation. But that this order was Peirce’s aim is nevertheless shown by his attempts to divest the oval of its truth-functional meaning. For example, in a ‘Memoir’ on EGs written for the National Academy of Science in December 1896, Peirce proposes to interpret the ovals as signs that merely highlight or quote what they encircle:

A lightly drawn endless line encircling a proposition or verb shall be a quotation mark; and whatever is vague in the encircled proposition shall be understood and designated (after the similar determination of everything vague in the parts of the graph outside the quotation mark has been declared), in such a way as to be favorable to the veracity of its author. And an undesignated author of a quoted proposition shall be understood to be the advocatus diaboli, or hypothetical indefinitely clever opponent of the assertion of the graph [26, p. 4].

That these boundary lines around graphs also signal negation comes from the polarity phenomenon: boundaries introduce orientation into the sheet of assertion. In semantic terms, when the polarity changes a role-switch is prompted between the utterer, or the one favorable to the veracity of the graph residing in the area of the cut, and the interpreter, or the opponent of the assertion of that graph. This is exactly what the meaning of (strong) negation is in game-theoretical semantics [17]. Syntactically, ovals need not be taken to assume the office of negations at all.

In another of the early pieces on logical graphs Peirce considers the function of the oval to be that of forming a subject of a meta-assertion and, in the absence of any meta-assertion, negating that subject:

\(^{15}\) See [38, pp. 8–9; 44, p. 19; 46, pp. 908–909]; see also [4].
The expression of a conceived fact being enclosed within a lightly drawn oval is thereby not asserted, but is looked upon as a subject of assertion. But if nothing is asserted of it, its being written enclosed on the sheet of the graph denies it [29, p. 1].

What becomes equated with the denial of the graph enclosed within the cut is the absence of any meta-assertion on the graph. This gives us the idea of a general convention regulating ovals that attributes to them, first and foremost, the collectional meaning. Any other truth-functional or non-truth-functional meaning is received only as a derived feature from conventions that supervene on general conventions.

What happens in Peirce’s logical graphs is a retraction of the proposition by the cut from the realm of substantive possibilities in which that proposition belongs. This retraction of the proposition from the sheet of assertion is the fact that makes the negated proposition unassertable. Such propositions that have cuts around them fail to actualize on the sheet of assertion, and in failing to do so they become denials. To put the matter in semantic terms, then, because the cut effects the role-change between the utterer (the verifier) and the interpreter (the falsifier) of the graph, the responsibility for demonstrating the truth of the assertion ceases to be requested. What takes its place is the task of showing the graph to be false, and this task is delivered to the opponent of the utterer. Since the opponent does not bear the responsibility for demonstrating the verity of the graph, that graph cannot itself be an assertion. It can only be the subject of some other, affirmative assertion. At once place, Peirce indeed remarks that the sheet of assertion be in fact termed the sheet of affirmation:

[W]hatever state of things you represent on this [recto] page, you will be understood to affirm as existing somewhere, or, at least, consistently to make believe to affirm [43].

What the negated propositions represent does not exist and thus cannot be affirmed to exist, either. What nonetheless can be asserted of the encircled proposition is a hypothetical, that ‘the effect of enclosing a proposition in an oval is to assert that were that proposition true some thing that is in fact false would be true’ [31].

In their standard formulation (e.g., [35]), Peirce’s logical graphs present a perfect coincidence of two signs and two operations. The sign of assertion and the sign of scope merge with the two truth-functional operations, conjunction and negation: the sign of assertion is also a sign of conjunction, and the sign of scope is also a sign of negation. Now, given this merger, are we to think that negation is for Peirce the polar opposite of assertion? Given his notation, are we to think that negation belongs to the illocutionary force of a sentence rather than to the thought expressed in it? The answer to this question is, again, negative. Like Frege, Peirce considers negation to be part of the locutionary content of a sentence and not part of the illocutionary force. In having both a truth-functional and a collectional meaning, the oval contributes both to the signification of the sentences expressed and to the syntactic structure of the sentence expressing that signification. In its truth-functional meaning, the oval determines truth-conditions, while in its collectional meaning it determines what it is whose truth-conditions are so determined. But neither truth-functional nor collectional meaning belong to what Frege calls the force with which a sentence is uttered. The negation in the standard formulation of Peirce’s Alpha and Beta graphs is not an illocutionary negation. Of the two necessary signs, assertion is illocutionary, grouping locutionary. In the logical graphs, conjunction and negation correspond to these signs, respectively, and conjunction can also be considered (according to the exportation test) as illocutionary, while negation cannot.

The graphical logic of existential graphs was not developed to deal with assertions as speech acts or other kinds of speech acts. Peirce does not say much about how the system could be extended or its rules modified to deal with various speech acts, and he developed most of his views on speech acts independently of his development of the logic of graphs, in which assertions are taken as responsibility-bearing actions of the utterers for the truth of the asserted propositions who can be sanctioned on uttering false assertions.

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In the remainder of this section, let us sketch a few lines of thoughts along which his graphical logic could be brought to bear on certain aspects of illocutionary logic.

For example, in order to create illocutionary negation in the graphical system, certain modifications to the diagrammatic syntax as well as to the behavior of the basic rules become inevitable. For example, certain novel tints or shadings of the areas of graphs along the lines in which Peirce proposed to extend the Gamma part of the system by tinctures\(^\text{16}\) may be added to represent performatives. Instead of the sheet of assertion, we could also have the ‘sheet of commission’, the ‘sheet of directive’, the ‘sheet of declaration’, and the ‘sheet of expression’, for example. Then the outer boundary of the respective tincture could be taken to stand for the illocutionary denial (such as in “I do not promise to \(p\)”). If in addition a propositional negation is present (as an oval or a cut), then the illocutionary act (“to promise”) stays on the area outside of the cut, which is marked on the inner boundary of the tincture and which represents the propositional negation (“I promise not to \(p\)”). Graphical illocutionary logic is thus not only conceivable but presents itself as a natural extension of tinctured graphs with cuts.

In illocutionary logic, the law of the excluded middle does not generally hold (to wit, stating “I do not promise to \(p\)” does not imply that I would at once be entitled also to assert that “I accept to \(p\)”). Thus in illocutionary logic the double-cut rule is not valid, and the cuts cannot occur around arbitrary propositions. The logical behavior of the resulting system is intuitionistic rather than classical. This poses no problem given that there exist such intuitionistic versions of Peirce’s logical graphs [19, 21]. Moreover, in the quantificational case, the graphical system for illocutionary speech acts would also likely be ‘non-Fregean’ in the sense that there will be complex lines (ligatures) that cross tinctures, which just as in modal graphs with quantifiers, but unlike in the standard non-modal cases, are themselves well-formed graphs.

Even without the tinctures and the two kinds of negations – one illocutionary at the outer boundary of the tincture and one contradictory at the inner boundary – the original broken-cut oval that Peirce introduced in 1903 to define systems of modal logic has a noticeable illocutionary flavor:

I will begin with one of the gamma cuts. I call it the broken cut. I scribe it thus

\[\text{It rains}\]

This does not assert that it does not rain. It only asserts that the alpha and beta rules do not compel me to admit that it rains, or what comes to the same thing, a person altogether ignorant, except that he was well versed in logic so far as it is embodied in the alpha and beta parts of existential graphs, would not know that it rained [34, p. 42; for\(\bigcirc\) in Peirce’s hand].

That is, the broken cut around a proposition \(p\) means the absence of any such compelling reasons from the extensional parts of the system (Alpha, Beta) to have one admit that \(p\). Modal propositions are assertions concerning states of information and uncertainty.

At all events, negation is not the primary function of the ovals even from the truth-functional point of view. In fact, Peirce is clear that the convention that the single oval signifies negation is derived from the convention that the double oval structure called ‘scroll’ signifies the conditional form. Negation is defined in Peirce’s logical graphs as the implication of the False in intuitionist logic. Its meaning is thus a derived meaning, and ‘corollary interpretation’ of the convention concerning the meaning of the scroll (material implication).\(^\text{17}\)

\(^\text{16}\) On tinctured graphs see [41, 42, 51, pp. 87–109].

\(^\text{17}\) See [4, 54, pp. 35–37; 59]. The derivation of negation from implication was already contained in [25].
The simple possibility of defining negation by implication and of having tinctured graphs for illocutionary logic has an immediate consequence for the question of whether negation occurs at the locutionary or also at the illocutionary level. The fact that the sign of illocutionary force is a necessary sign of the system implies that the sign of whatever illocutionary force is necessary, that is, that it is always necessary to indicate the illocutionary force. That this is so in Peirce’s tinctured graphs has been shown above. But the negation sign is definable, via material implication, and thus is not at all necessary. Therefore, the negation sign is not in general a sign of illocutionary force.

3. Conclusions

We presented five views about propositions, assertions and other kinds of speech acts.

(1) Different speech-act types (illocutionary acts) may have the same propositional content (Peirce’s Kanál στοχεία observation)

(2) The position of a proposition in discourse (the way a proposition appears in discourse) determines whether it is used assertively or non-assertively (the Peano Point)

(3) The distinction between assertive and non-assertive occurrences of a proposition must be notationally expressed, that is, must be observed from the notation used, whatever form the notational expression of this difference happens to take (the Dudman Point)

(4) A good notation should indicate assertive occurrences of a proposition by an independent, primitive sign of assertion (the Frege Point)

(5) A proposition may occur in discourse now asserted, now unasserted (the Geach Point)

We have seen that (1) implies (5), that (2) and (3) imply (or, rather, presuppose) (5), that (4) implies (3) and (5) but is not implied by (2), (3) or (5), and that (5) does not imply any of (1)–(4). If (3) is regarded as a normative requirement for a good notation, then we can also state that (5) normatively implies (3). We also saw that Peirce endorsed the points by Peano (2), Dudman (3) and Geach (5), but not that of Frege (4).

Frege has an ad hoc sign of assertion and an ad hoc sign of negation. Peirce has a sign of assertion which is also a sign of logical conjunction, and a sign of scope which is also a sign of negation (and of other logical notions in different versions of the logical graphs). The motivation for Peirce’s merger of certain notational functions (asserting, grouping) with certain truth-functions (negation, conjunction, etc.) is, at bottom, the same as Frege’s, namely to be able to carry logical analysis as far as possible. And if there is a notation in which the sign of a truth-functional operation can be at the same time the sign of its own scope, this has to be done for analysis’ sake.

As with Frege, Peirce’s denial is not the polar opposite of assertion and not part of the illocutionary force: the denial of the assertion of p is not a non-assertion of p but an assertion of not-p. This is reflected in the notation of logical graphs, where the oval is primarily a sign of grouping, and secondarily, and not even necessarily, a sign of negation. If denial were the polar opposite to assertion, this should, on Fregean principles, be shown in the notation by an ad hoc sign of denial. But since it is not, then Frege can and does dispense with a separately definable sign of denial. But Peirce’s point is a different, notational point: the fact that the sign of negation, being primarily a sign of scope, can also be a sign of something else, is a notational demonstration that negation only occurs at the locutionary level. If it occurred at the illocutionary level, it would be a necessary sign of the notation. But negation is not a necessary sign of the notation, because it is definable and thus eliminable. Therefore, negation cannot generally occur at the illocutionary level.

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18 We are indebted to a reviewer of this journal for the suggestion of presenting this summary of the main points and their relationships.

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