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HIGH-ORDER METHODS FOR VOLterra INTEGRAL EQUATIONS WITH GENERAL WEAK SINGULARITIES

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A piecewise polynomial collocation method for solving linear Volterra integral equations of the second kind is constructed. The kernel of the integral equation may have weak diagonal and boundary singularities. Using suitable smoothing techniques and polynomial splines on mildly graded or uniform grids, optimal global convergence estimates are derived and a local superconvergence result is proven.

Keywords Smoothing transformation; Spline collocation method; Volterra integral equation; Weakly singular kernel.

AMS Subject Classification 45D05; 65R20.

1. INTRODUCTION

Let \( R = (-\infty, \infty), \mathbb{N} = \{1, 2, \ldots \}, \mathbb{N}_0 = \{0\} \cup \mathbb{N}, \)
\( D_b = \{(x, y) : 0 < y < x \leq b\}, \quad D'_b = \{(x, y) : 0 < y \leq x \leq b\}. \)

In the present article we shall focus our attention on equations

\[ u(x) = f(x) + \int_0^x K(x, y)u(y)dy, \quad 0 \leq x \leq b, \quad (1.1) \]

with kernels of the form

\[ K(x, y) = K_1(x, y)(x - y)^{-\tau} + K_2(x, y), \quad (x, y) \in D_b, \quad 0 < \tau < 1, \quad (1.2) \]
where \( K_1 \) and \( K_2 \) are some \( n \) times \((n \geq 0)\) continuously differentiable functions on \( D'_b \) such that

\[
\left| \left( \frac{\partial}{\partial x} \right)^i \left( \frac{\partial}{\partial y} \right)^j K_l(x, y) \right| \leq cy^{\lambda_i - \lambda_j}, \quad (x, y) \in D'_b, \quad l = 1, 2, \tag{1.3}
\]

\[
i, j \in \mathbb{N}_0, \quad i + j \leq n, \quad \lambda_1, \lambda_2 \in (-\infty, 1), \quad \lambda_1 < 1 - v. \tag{1.4}
\]

In (1.3) \( c \) is a constant (below by \( c, c_1, \ldots \) will be denoted positive constants that may have different values by different occurrences).

An example of a function \( K \) satisfying (1.2)–(1.4) is given by the formula

\[
K(x, y) = K_1(x, y)y^{-\lambda_1}(x - y)^{-v} + K_2(x, y)y^{-\lambda_2},
\]

where \( 0 < v < 1, \lambda_1 < 1 - v, \lambda_2 < 1 \) and \( K_1, K_2 \in C^n(D_b), \quad D_b = \{(x, y) : 0 \leq y \leq x \leq b\} \). Here and below by \( C^n(\Omega) \) we denote the set of \( n \) times continuously differentiable functions on \( \Omega, \ C^n(\Omega) = C(\Omega) \).

The numerical solution of Volterra integral equations has been considered by many authors. The survey article [2] and the monographs [5, 8] give a good picture of these developments and contain an extensive bibliography, see also some recent works [9, 12, 17]. For Volterra equations (1.1) with bounded kernels, the smoothness of the kernel \( K \) and the forcing function \( f \) determine the smoothness of the solution \( u \) on the entire interval \([0, b]\). If we allow Abel-type kernels of the form \( K(x, y) = K_1(x, y)(x - y)^{-v}, \quad 0 < v < 1, \) where \( K_1(x, y) \) is sufficiently smooth for \((x, y) \in D_b\), then the resulting solutions are typically nonsmooth at the boundary point \( 0 \) of the interval \([0, b]\), where their derivatives become unbounded, see [5, 6, 8] and Lemma 2.3. In collocation methods the singular behavior of the exact solution can be taken into account by using polynomial splines on special nonuniform grids that are properly graded in order to compensate the generic boundary singularities of the derivatives of the exact solution. We refer to [4–6, 8] for details; see also [1, 7, 18, 24]. However, in practice, the use of strongly graded grids may cause serious implementation problems since such grids may create significant round-off errors in calculations and therefore lead to unstable behavior of numerical results.

To avoid problems associated with the use of strongly graded grids we first perform in Equation (1.1) a change of variables so that the singularities of the derivatives of the exact solution will be milder or disappear and after that we solve the transformed equation by a collocation method on the mildly graded or uniform grid. Our approach is based on the ideas and results of [21, 22]; see also [3, 10, 11, 15, 16, 20, 23, 25, 26]. The main purpose of the present article is to extend the corresponding
results of [15, 20, 21] to a wider class of equations and to study the attainable order of global and local convergence of proposed algorithms. The main results of the article are given by Theorems 5.4 and 6.1.

2. REGULARITY OF THE SOLUTION

For given $m \in \mathbb{N}$, $\theta \in \mathbb{R}$, $\theta < 1$, let $C^{m,\theta}(0, b]$ be the Banach space of functions $z \in C[0, b] \cap C^m(0, b]$ such that

$$
\|z\|_{C^{m,\theta}(0, b]} \equiv \sum_{k=0}^{m} \sup_{0 < x \leq b} w_{k-1+\theta}(x) |z^{(k)}(x)| < \infty.
$$

Here

$$w_{\beta}(\beta) = \begin{cases}
1 & \text{for } \beta < 0 \\
1/(1 + |\log \beta|) & \text{for } \beta = 0 \\
\beta^x & \text{for } \beta > 0
\end{cases}, \quad x, \beta \in \mathbb{R}, \ \beta > 0.
$$

Clearly, $C^m[0, b] \subset C^{m,\theta}(0, b] \subset C^{m,\theta_1}(0, b]$ for $\theta < \theta_1 < 1$. Let $T$ be the integral operator of Equation (1.1),

$$(Tu)(x) = \int_0^x K(x, y)u(y)dy, \quad 0 \leq x \leq b.
$$

Lemma 2.1. Assume (1.2)–(1.4) with $n = 0$. Then $T$ maps $L^\infty(0, b)$ into $C[0, b]$ and $T : L^\infty(0, b) \to C[0, b]$ is compact.

The proof can be built using the Arzelà theorem, cf. [22].

If $f \in C[0, b]$, then it follows from Lemma 2.1 that Equation (1.1) has a unique solution $u \in C[0, b]$.

Lemma 2.2. Assume (1.2)–(1.4) with $n \in \mathbb{N}$ and let $\lambda_2 < \nu + \lambda_1$. Then $T$ maps $C^{n,\theta}(0, b]$ with $\theta = \nu + \lambda_1$ into $C^{n,\theta}(0, b]$ and $T : C^{n,\theta}(0, b] \to C^{n,\theta}(0, b]$ is compact.

Proof. For any $(x, y) \in D_0$, $k, l \in \mathbb{N}_0$, $k + l \leq n$, we find by (1.3) that

$$
\left| \frac{\partial^k}{\partial x^k} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^l [K_1(x, y)(x - y)^{-\nu}] \right|
\quad = \left| \frac{\partial^k}{\partial x^k} \left[ (x - y)^{-\nu} \sum_{j=0}^{l} \binom{l}{j} \left( \frac{\partial}{\partial x} \right)^{l-j} \left( \frac{\partial}{\partial y} \right)^{j} K_1(x, y) \right] \right|
$$
Lemma 2.3. Assume
\[ 0 = c(x - y)^{-\nu} \sum_{i=0}^{k} \binom{k}{i} \sum_{j=0}^{l} \left( \frac{\partial}{\partial x} \right)^{i} \left( \frac{\partial}{\partial y} \right)^{j} K_1(x, y) \]
\[ \leq c_1(x - y)^{-\nu} y^{-\lambda_1 - \ell} \]
and
\[ \left| \left( \frac{\partial}{\partial x} \right)^{k} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)^{l} K_2(x, y) \right| \leq c y^{-\lambda_2 - \ell} \leq c(x - y)^{-\nu} y^{-\lambda_2 - \ell}, \]
where \( 0 < \nu < 1 \). Since \( \lambda_2 < \nu + \lambda_1 \), we can fix \( \nu' \) so that \( \nu' + \lambda_2 \leq \nu + \lambda_1 \). According to the results of [22], inequalities (2.1) and (2.2) with \( \nu' + \lambda_2 \leq \nu + \lambda_1 \) guarantee that \( T \) maps \( C^{\nu, \beta}(0, b) \) into \( C^{\nu, \beta}(0, b) \) and \( T : C^{\nu, \beta}(0, b) \to C^{\nu, \beta}(0, b) \) is compact for any \( \theta \) satisfying \( \nu + \lambda_1 \leq \theta < 1 \). In particular, the assertion of Lemma 2.2 holds. \( \Box \)

The following lemma is a consequence of Lemma 2.2.

Lemma 2.3. Assume (1.2)–(1.4) for \( n \in \mathbb{N} \). Let \( \lambda_2 < \nu + \lambda_1 \) and \( f \in C^{n, \nu_2 + \lambda_1}(0, b) \). Then Equation (1.1) is uniquely solvable and its solution belongs to \( C^{n, \nu_2 + \lambda_1}(0, b) \).

3. SMOOTHING TRANSFORMATION

For given \( \varrho \in [1, \infty) \) denote
\[ \varphi(s) = b^{1-\varrho} s^\varrho, \quad 1 \leq \varrho < \infty, \quad 0 \leq s \leq b. \]
(3.1)

Clearly, \( \varphi \in C[0, b] \), \( \varphi(0) = 0 \), \( \varphi(b) = b \) and \( \varphi'(s) > 0 \) for \( 0 < s \leq b \). Thus, \( \varphi \) maps \([0, b]\) onto \([0, b]\) and has a continuous inverse \( \varphi^{-1} : [0, b] \to [0, b] \) with \( \varphi^{-1}(0) = 0 \) and \( \varphi^{-1}(b) = b \). Note that \( \varphi(s) = s \) for \( \varrho = 1 \). We are interested in (3.1) with \( \varrho > 1 \) since then (3.1) possesses a smoothing property for \( u(\varphi(s)) \) with singularities of the derivatives of \( u(x) \) at \( x = 0 \):

Lemma 3.1. Let \( u \in C^{m, \beta}(0, b) \), \( m \in \mathbb{N} \), \( -\infty < \theta < 1 \), and let \( \varphi \) be given by formula (3.1) where \( \varrho \in \mathbb{N} \) for \( \varrho \leq m \) and \( \varrho \in \mathbb{R} \) for \( \varrho > m \). Furthermore, let \( v(s) = u(\varphi(s)) \), \( 0 \leq s \leq b \). Then \( v \in C^{m, \beta}(0, b) \), where \( \theta_\varrho = 1 - \varrho(1 - \theta) \).

Proof. See [21]. \( \Box \)

Remark 3.2. Instead of (3.1) other smoothing transformations can be used. We refer to [21] for a general discussion in this connection.
4. PIECEWISE POLYNOMIAL INTERPOLATION

For given \(m, N \in \mathbb{N}\), let

\[
\mathcal{S}_{m-1}^{(-1)}(\Delta_N') = \{v_N : v_N|_{[t_{j-1}, t_j]} \in \pi_{m-1}, j = 1, \ldots, N\} \quad (4.1)
\]

be the spline space of piecewise polynomial functions on the grid

\[
\Delta_N' = \{t_0, \ldots, t_N : 0 = t_0 < \cdots < t_N = b\}
\]

with the nodes

\[
t_j = b(j/N)', \quad j = 0, \ldots, N, \quad N \in \mathbb{N}, \quad r \in \mathbb{R}, \quad r \geq 1.
\] (4.2)

In (4.1) \(v_N|_{[t_{j-1}, t_j]} (j = 1, \ldots, N)\) is the restriction of \(v_N(t), t \in [0, b]\), to the subinterval \([t_{j-1}, t_j] \subset [0, b]\) and \(\pi_{m-1}\) denotes the set of all polynomials of degree not exceeding \(m - 1\). Note that the elements of \(\mathcal{S}_{m-1}^{(-1)}(\Delta_N')\) may have jump discontinuities at the interior points \(t_1, \ldots, t_{N-1}\) of the grid \(\Delta_N'\).

In every subinterval \([t_{j-1}, t_j] (j = 1, \ldots, N)\) we introduce \(m\) interpolation points \(t_{j1}, \ldots, t_{jm}\) as follows:

\[
t_{jl} = t_{j-1} + \eta_l(t_j - t_{j-1}), \quad l = 1, \ldots, m; \quad j = 1, \ldots, N,
\] (4.3)

where \(\eta_1, \ldots, \eta_m\) are some fixed parameters such that

\[
0 \leq \eta_1 < \cdots < \eta_m \leq 1.
\] (4.4)

Let \(P_N^{(m-1)} (m, N \in \mathbb{N})\) be an operator that assigns to every continuous function \(z \in C[0, b]\) a piecewise polynomial function \(P_N^{(m-1)}z \in \mathcal{S}_{m-1}^{(-1)}(\Delta_N')\) such that \((P_N^{(m-1)}z)(t_{jl}) = z(t_{jl}), l = 1, \ldots, m; \ j = 1, \ldots, N\). As easily seen (cf. [24]) the norms of \(P_N^{(m-1)} : C[0, b] \to L^\infty(0, b)\) are bounded by a constant \(c = c(m)\) that is independent of \(N\),

\[
\|P_N^{(m-1)}\|_{C[0, b] \to L^\infty(0, b)} \leq c, \quad N \in \mathbb{N},
\] (4.5)

and

\[
\|z - P_N^{(m-1)}z\|_{L^\infty(0, b)} \to 0 \quad \text{as} \ N \to \infty \quad \text{for every} \ z \in C[0, b].
\] (4.6)

**Lemma 4.1.** Let \(z \in C^{m, \beta}(0, b), \ m \in \mathbb{N}, \ -\infty < \theta < 1\). Then for every \(j = 1, \ldots, N\),

\[
\sup_{t_{j-1} \leq t \leq t_j} |z(t) - (P_N^{(m-1)}z)(t)| \leq c(t_j - t_{j-1})^m \begin{cases} 1 & \text{for} \ m < 1 - \theta, \\ 1 + |\log t_j| & \text{for} \ m = 1 - \theta, \\ t_j^{1-\theta-m} & \text{for} \ m > 1 - \theta, \end{cases}
\]
and
\[ \| z - P_N^{(m-1)} z \| \leq c \varepsilon^{(m,r)}_N. \]

Here \( c > 0 \) is a constant not depending on \( j \) and \( N \) and
\[ \varepsilon^{(m,r)}_N = \begin{cases} 
N^{-m} & \text{for } m < 1 - \theta, \ r \geq 1, \\
N^{-m}(1 + \log N) & \text{for } m = 1 - \theta, \ r = 1, \\
N^{-m} & \text{for } m = 1 - \theta, \ r > 1, \\
N^{-r(1-\theta)} & \text{for } m > 1 - \theta, \ 1 \leq r < \frac{m}{1-\theta}, \\
N^{-m} & \text{for } m > 1 - \theta, \ r \geq \frac{m}{1-\theta}, \ r \geq 1. 
\end{cases} \tag{4.7} 
\]

Proof. See [6, 24]. \( \Box \)

5. COLLOCATION METHOD

We introduce in (1.1) the change of variables \( y = \varphi(s), x = \varphi(t), s, t \in [0, b] \). We obtain an integral equation of the form
\[ v(t) = \int_0^t K_\varphi(t, s)v(s)\, ds + f_\varphi(t), \quad 0 \leq t \leq b, \tag{5.1} \]
where
\[ K_\varphi(t, s) = K(\varphi(t), \varphi(s))\varphi'(s), \quad f_\varphi(t) = f(\varphi(t)) \tag{5.2} \]
are given functions and \( v(t) = u(\varphi(t)) \) is a function that we have to find.

We find an approximation \( v_N = v_{N,\varphi,m,r} \) to \( v \), the solution to (5.1), from the following conditions:
\[ v_N \in S_{m-1}^{-1}(\Delta_N'), \quad N, m \in \mathbb{N}, \ r \in [1, \infty), \tag{5.3} \]
\[ v_N(t_{lj}) = \int_0^{t_{lj}} K_\varphi(t_{lj}, s)v_N(s)\, ds + f_\varphi(t_{lj}), \quad l = 1, \ldots, m; \ j = 1, \ldots, N, \tag{5.4} \]
with \( \{t_{lj}\} \), given by (4.3). Having found the approximation \( v_N \) to \( v \), we determine an approximation \( u_N = u_{N,\varphi,m,r} \) for \( u \), the solution to (1.1), setting
\[ u_N(x) = v_N(\varphi^{-1}(x)), \quad 0 \leq x \leq b. \tag{5.5} \]
Remark 5.1. The choice of nodes (4.3) with \( \eta_1 = 0, \eta_m = 1 \) (see (4.4)) actually implies that the resulting collocation approximation \( u_N \) belongs to the smoother polynomial spline space \( S_{m-1}^{(0)}(\Delta^v) = \{v_N \in C[0, b] : v_N|_{\eta_{j-1}, \eta_j} \in \pi_{m-1}, j = 1, \ldots, N \} \).

Remark 5.2. The conditions (5.3)–(5.4) form a system of equations whose exact form is determined by the choice of a basis in \( S_{m-1}^{(v)}(\Delta^v) \) (or in \( S_{m-1}^{(0)}(\Delta^v) \) if \( \eta_1 = 0, \eta_m = 1 \)). We refer to [21] for a suitable choice of it.

Theorem 5.3. Let \( f \in C[0, b] \) and assume (1.2)–(1.4) with \( n = 0 \). Furthermore, assume that \( \varphi \) is given by formula (5.1) and the interpolation nodes (4.3) with grid points (4.2) and parameters (4.4) are used.

Then (1.1) has a unique solution \( u \in C[0, b] \), the settings (5.3)–(5.5) determine for sufficiently large values of \( N \), say \( N \geq N_0 \), a unique approximation \( u_N \) to \( u \), and

\[
\|u_N - u\|_\infty \equiv \sup_{0 \leq x \leq b} |u_N(x) - u(x)| \to 0 \quad \text{as} \quad N \to \infty. \tag{5.6}
\]

Proof. We write (5.1) in the form \( v = T_\varphi v + f_\varphi \) where \( T_\varphi \) is defined by formula

\[
(T_\varphi v)(t) = \int_0^t K_\varphi(t, s)v(s)ds, \quad 0 \leq t \leq b. \tag{5.7}
\]

It follows from (1.2)–(1.4) and (3.1) that \( K_\varphi(t, s) = K(\varphi(t), \varphi(s))\varphi'(s) \) is a continuous function for \( (t, s) \in D_b \) and

\[
|K_\varphi(t, s)| \leq c[(t-s)^{-\gamma_1} + s^{-\gamma_2}], \quad (t, s) \in D_b. \tag{5.8}
\]

Therefore, by Lemma 2.1, \( T_\varphi \) is compact as an operator from \( L^\infty(0, b) \) into \( C[0, b] \). This together with \( f_\varphi \in C[0, b] \) implies that equation \( v = T_\varphi v + f_\varphi \) has a unique solution \( v = (I - T_\varphi)^{-1}f_\varphi \in C[0, 1] \). Here \( I \) is the identity mapping and \( (I - T_\varphi)^{-1} \) is bounded as an operator from \( L^\infty(0, b) \) into \( C[0, b] \).

Further, conditions (5.3) and (5.4) have the operator equation representation

\[
v_N = P_N^{(m-1)}T_\varphi v_N + P_N^{(m-1)}f. \tag{5.9}
\]

Using a standard arguing (cf. e.g., [21, 24]) we obtain that \( I - P_N^{(m-1)}T_\varphi \) is invertible for all \( N \geq N_0 \) and the norms of \( (I - P_N^{(m-1)}T_\varphi)^{-1} \) are uniformly bounded:

\[
\|(I - P_N^{(m-1)}T_\varphi)^{-1}\|_{L^\infty(0, b) \to L^\infty(0, b)} \leq c, \quad N \geq N_0, \tag{5.10}
\]
with a constant $c > 0$ that is independent of $N$. Thus, Equation (5.9) has for sufficiently large $N$ a unique solution $v_N \in S_{m-1}^{r-1}(\Delta_N)$. We have for it and $v$, the solution of equation $v = T_x v + f_x$, that

$$v_N - v = (I - P_{N}^{(m-1)} T_x)^{-1} (P_{N}^{(m-1)} v - v).$$

Therefore, due to (5.10),

$$\|v_N - v\|_\infty \leq c \|v - P_{N}^{(m-1)} v\|_\infty, \quad N \geq N_0,$$

(5.11)

where $c$ is a positive constant not depending on $N$. Using $v(t) = u(\varphi(t))$ and (5.5), we get

$$\|u_N - u\|_\infty = \|v_N - v\|_\infty.$$  \hspace{1cm} (5.12)

This together with $v \in C[0, b]$, (4.6) and (5.11) yield (5.6). $\square$

**Theorem 5.4.** For given $m \in \mathbb{N}$ assume (1.2)–(1.4) with $n = m$. Let $\lambda_2 < v + \lambda_1$, $f \in C^{n, v+\lambda_1}(0, b]$, and let $\varphi$ be given by formula (3.1) where $\varrho \in \mathbb{N}$ for $\varrho \leq m$ and $\varrho \in \mathbb{R}$ for $\varrho > m$. Finally, let the interpolation nodes (4.3) with grid points (4.2) and parameters (4.4) be used.

Then the settings (5.3)–(5.5) determine for $N \geq N_0$ a unique approximation $u_N$ to $u$, the solution of Equation (1.1), and

$$\|u_N - u\|_\infty = \|v_N - v\|_\infty \leq c \begin{cases} N^{-m} & \text{for } m < \varrho(1 - v - \lambda_1), \ r \geq 1, \\ N^{-\varrho}(1 + \log N) & \text{for } m = \varrho(1 - v - \lambda_1), \ r = 1, \\ N^{-m} & \text{for } m = \varrho(1 - v - \lambda_1), \ r > 1, \\ N^{-r\varrho(1-v-\lambda_1)} & \text{for } m > \varrho(1 - v - \lambda_1), \ 1 \leq r < \frac{m}{\varrho(1 - v - \lambda_1)}, \\ N^{-m} & \text{for } m > \varrho(1 - v - \lambda_1), \ r \geq \frac{m}{\varrho(1 - v - \lambda_1)}. \end{cases}$$  \hspace{1cm} (5.13)

Here $c$ is a positive constant not depending on $N$.

**Proof.** On the basis of Lemmas 2.3 and 3.1 we find that $v \in C^{n, \tilde{\varrho}}(0, b]$, with $\tilde{\varrho} = 1 - \varrho(1 - v - \lambda_1)$. Therefore, by Lemma 4.1,

$$\|v - P_{N}^{(m-1)} v\|_\infty \leq c \varepsilon_N^{(m, \tilde{\varrho}, r)},$$

with a positive constant $c$ that is independent of $N$. This together with (5.11) and (5.12) yields (5.13). $\square$
Remark 5.5. For \( p = 1 \) (i.e., \( \varphi(s) \equiv s \), see (3.1)), Theorem 5.4 establishes the order of global convergence of a piecewise polynomial collocation method applied directly (without any change of variables) to the integral Equation (1.1).

Remark 5.6. It follows from Theorem 5.4 that the accuracy \( \| u_N - u \|_{\infty} = \| v_N - v \|_{\infty} \leq cN^{-m} \) can be achieved on a mildly graded or uniform grid. For example, if \( \nu = 0.3, \lambda_1 = 0.2, \lambda_2 = 0.49, m = 3 \) (the case of quadratic polynomials), \( q > 6 \), then the maximal convergence order \( \| u_N - u \|_{\infty} \leq cN^{-3} \) is available for \( r \geq 1 \). In particular, the uniform grid \((r = 1\) in (4.2)) may be used.

Remark 5.7. Theorem 5.4 proposes, in particular, how \( r \) and \( \varrho \) should be chosen to achieve by splines of degree \( m - 1 \) the highest possible convergence order \( \| v_N - v \|_{\infty} \leq cN^{-m} \). Nevertheless, as shown in Theorem 6.1, the attainable order of local convergence may be \( O(N^{-(m+\beta)}) \) for some \( \beta = \beta(\nu) > 0 \) by a special choice of parameters \( \eta_1, \ldots, \eta_m \) (see (4.3), (4.4)).

6. SUPERCONVERGENCE RESULTS

In this section we show that, in addition to Theorem 5.4, assuming some additional smoothness of \( K \) and \( f \) and choosing more carefully the collocation parameters (4.4) in method (5.3)–(5.5), the superconvergence of \( v_N \) at the collocation points (4.3) can be established, cf. [5–8, 13, 14, 21]. In particular, on the conditions of Theorem 6.1 below we get

\[
\max_{l=1,\ldots,m; j=1,\ldots,N} |v_N(t_{jl}) - v(t_{jl})| = \max_{l=1,\ldots,m; j=1,\ldots,N} |u_N(\varphi(t_{jl})) - u(\varphi(t_{jl}))| \leq cN^{-m-(1-v)},
\]

provided that \( 2rg(1-v-\lambda_1) \geq m + 1 - v \).

As usual (see, e.g., [1, 6, 19, 24]), the key for the proof of the superconvergence results is a detailed study of the norm \( \| T_\vartheta(P_N^{(m-1)}v - v) \|_{\infty} \) where \( v \) is the solution to Equation (5.1). The estimation of this norm is based upon the regularity properties of \( v \) and some ideas of [24] for a suitable expansion of \( T_\vartheta(P_N^{(m-1)}v - v) \). Since the kernel \( K(x,y) \) in our case, in addition to a diagonal singularity at \( x = y \), may also have a boundary singularity as \( y \to 0 \), we have to overcome some additional difficulties, by computing and estimating the integrals occurring in the collocation systems; about the case of standard weakly singular kernels, without a boundary singularity see, for example, [5, 7, 8, 19]. Note that for \( 0 < v < 1, \lambda_1 = 0, K_\vartheta(x,y) \equiv 0 \) and \( \rho = 1 \) Theorem 6.1 strengthens the known results about the superconvergence in the sense that the corresponding values of \( r \) can be taken smaller than those proposed in [6].
Theorem 6.1. For given \( m \in \mathbb{N} \), assume (1.2)–(1.4) with \( n = m + 1 \) and let \( \lambda_2 < \nu + \lambda_1 \). Let \( f \in C^{m+1,\nu+\lambda_1}(0, b] \) and let \( \varphi \) be given by formula (3.1) where \( \varrho \in \mathbb{N} \) for \( \varrho \leq m + 1 \) and \( \varrho \in \mathbb{R} \) for \( \varrho > m + 1 \). Furthermore, let the interpolation nodes (4.3) be generated by the grid points (4.2) and by the node points \( \eta_1, \ldots, \eta_m \) of a quadrature approximation

\[
\int_0^1 z(s) ds \approx \sum_{l=1}^m w_l z(\eta_l), \quad 0 \leq \eta_1 < \cdots < \eta_m \leq 1, \tag{6.1}
\]

which, with appropriate weights \( \{w_l\} \), is exact for all polynomials of degree not exceeding \( m \).

Then for all sufficiently large \( N \in \mathbb{N} \), \( N \geq N_0 \), method (5.3)–(5.5) determines a unique approximation \( u_N \) to \( u \), the solution of Equation (1.1), and

\[
\max_{l=1, \ldots, m; \ j=1, \ldots, N} \left| u_N(\varphi(t_j)) - u(\varphi(t_j)) \right| \leq c E_N^{(m, \nu, \lambda_1, \varrho, r)}. \tag{6.2}
\]

Here \( t_j \) (\( l = 1, \ldots, m; \ j = 1, \ldots, N \)) are defined by (4.3), \( c \) is a positive constant not depending on \( N \), and

\[
E_N^{(m, \nu, \lambda_1, \varrho, r)} = \begin{cases} N^{-2\varrho(1-\nu-\lambda_1)} & \text{for } 1 \leq r < \frac{m+1-\nu}{2\varrho(1-\nu-\lambda_1)}, \\ N^{-m-1+\nu} & \text{for } r \geq \frac{m+1-\nu}{2\varrho(1-\nu-\lambda_1)}, \ r \geq 1. \end{cases} \tag{6.3}
\]

Proof. We know from the proof of Theorem 5.3 that Equation (5.9) has a unique solution \( v_N \) for \( N \geq N_0 \). We have for it and \( v \), the solution to (5.1), that

\[
(I - P_N^{(m-1)} T_\varphi)(v_N - P_N^{(m-1)} v) = P_N^{(m-1)} T_\varphi(P_N^{(m-1)} v - v), \quad N \geq N_0.
\]

This together with (4.5) and (5.10) yields

\[
\| v_N - P_N^{(m-1)} v \|_\infty \leq c \| T_\varphi(P_N^{(m-1)} v - v) \|_\infty, \quad N \geq N_0,
\]

with a constant \( c > 0 \) independent of \( N \). Further, let \( u \) be the solution to (1.1) and \( u_N \) be the approximation for \( u \), determined by (5.5). Then

\[
|u_N(\varphi(t_j)) - u(\varphi(t_j))| = |v_N(t_j) - v(t_j)| = \left| v_N(t_j) - (P_N^{(m-1)} v)(t_j) \right| \leq \| v_N - P_N^{(m-1)} v \|_\infty, \quad l = 1, \ldots, m; \ j = 1, \ldots, N.
\]

Therefore,

\[
\max_{l=1, \ldots, m; \ j=1, \ldots, N} |u_N(\varphi(t_j)) - u(\varphi(t_j))| \leq c \| T_\varphi(P_N^{(m-1)} v - v) \|_\infty, \tag{6.4}
\]
where
\[
\| T_\tau (P_N^{(m-1)} v - v) \|_\infty = \sup_{0 \leq t \leq b} \left| \int_0^\tau K_\tau(t,s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \\
\leq \sup_{0 \leq t \leq \bar{t}} |S(t)| + \sup_{\bar{t} \leq s \leq b} |S(s)|,
\]
with
\[
S(t) = \int_0^\tau K_\tau(t,s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds, \quad 0 \leq t \leq b.
\]
Since \( v(s) = u(\varphi(s)) \) and \( u \) is the solution of Equation (1.1), it follows from Lemmas 2.3 and 3.1 that
\[
v \in C^{m+\tilde{\beta}}(0,b) \subset C^{m\tilde{\beta}}(0,b), \quad \tilde{\beta} = 1 - \varrho(1 - v - \lambda_1).
\]
Let \( 0 \leq t \leq b \), \( \tau = \min\{t, \bar{t}\} \). Due to (1.2)–(1.4), (3.1), and (5.2) we get
\[
\int_0^\tau |K_\tau(t,s)| ds \leq c \int_0^\tau \left[ s^{-\varrho_1} (\tau^\rho - s^\rho)^{-\rho} + s^{-\varrho_2} \right] ds \\
\leq c_1 \left[ t^{(\varrho(1-v) - \lambda_1)} \int_0^\tau s^{-\lambda_1} (\tau - s)^{-\rho} ds + \int_0^\tau s^{-\varrho_2 + \rho - 1} ds \right] \\
\leq c_2 \left[ t^{(1-v) - \lambda_1} + t^{\varrho(1-\lambda_2)} \right] \\
\leq c_3 \left[ N^{-\rho q(1-v) - \lambda_1} + N^{-\varrho q(1-\lambda_2)} \right] \leq c_4 N^{-\rho q(1-v) - \lambda_1}.
\]
This together with Lemma 4.1 yields
\[
\left| \int_0^\tau K_\tau(t,s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \\
\leq c \left( \sup_{0 \leq s \leq \bar{t}} \left| (P_N^{(m-1)} v)(s) - v(s) \right| \right) \int_0^\tau |K_\tau(t,s)| ds \\
\leq \begin{cases} 
N^{-\rho q(1-v) - \lambda_1} + m & \text{for } m < \varrho(1 - v - \lambda_1) \\
N^{-\rho q(1-v) - \lambda_1} + m \log N & \text{for } m = \varrho(1 - v - \lambda_1) \\
N^{-2\rho(1-v) - \lambda_1} & \text{for } m > \varrho(1 - v - \lambda_1) 
\end{cases}
\]
\[
\leq \begin{cases} 
N^{-m-1+r} & \text{for } m \leq \varrho(1 - v - \lambda_1), \ r \geq 1, \\
N^{-2\rho(1-v) - \lambda_1} & \text{for } m > \varrho(1 - v - \lambda_1), \ 1 \leq r < \frac{m+1-v}{2\varrho(1-v-\lambda_1)}, \\
N^{-m-1+r} & \text{for } m > \varrho(1 - v - \lambda_1), \ r \geq \frac{m+1-v}{2\varrho(1-v-\lambda_1)}, \ r \geq 1. 
\end{cases}
\]
Thus, we have for any $t \in [0, b]$ that
\[
\left| \int_0^{\min\{t, N\}} K_\psi(t, s) \left( (P_N^{m-1})^v(s) - v(s) \right) ds \right| \\
\leq c \begin{cases} 
N^{-2q(1-v, \lambda_1)} & \text{for } 1 \leq r < \frac{m+1-v}{2q(1-v, \lambda_1)}, \\
N^{-m-(1-v)} & \text{for } r \geq \frac{m+1-v}{2q(1-v, \lambda_1)}, \quad r \geq 1.
\end{cases} 
\tag{6.7}
\]
In particular,
\[
\sup_{0 \leq t \leq t_1} |S(t)| \leq c \mathcal{E}_N^{(m, v, \lambda_1, q, r)}, 
\tag{6.8}
\]
with a constant $c > 0$ which is independent of $N$.

Further, let $t_{j-1} \leq t \leq t_j$, $2 \leq j \leq N$. Then, using (1.2)–(1.4), (3.1), (5.2), and $t_{j-1} \geq 2 t_j$, we obtain
\[
\int_{t_{j-1}}^t |K_\psi(t, s)| ds \leq c \int_{t_{j-1}}^t \left[ s^{-\lambda_2} \left( t^\theta - s^\theta \right) + s^{\lambda_2} \right] s^{-1} ds \\
\leq c_1 \left[ t^{(1-v, \lambda_1)} \int_{t_{j-1}}^t s^{-\lambda_1}(t-s)^{-v} ds + \int_{t_{j-1}}^t s^{\lambda_1-1} ds \right] \\
\leq c_2 \left[ t^{(1-v, \lambda_1)-1+v} (t_j - t_{j-1})^{1-v} + t_j^{(1-v, \lambda_1)-1} (t_j - t_{j-1}) \right].
\]
Since $\lambda_2 < v + \lambda_1$ and
\[
0 < (t_j - t_{j-1})^\alpha \leq c N^{-r} j^{(1-v, \lambda_1)}, \quad \alpha > 0, \quad j = 1, \ldots, N,
\]
we get for any $t \in [t_{j-1}, t_j]$ ($2 \leq j \leq N$) that
\[
(t_j - t_{j-1})^\gamma \int_{t_{j-1}}^t |K_\psi(t, s)| ds \\
\leq c \left[ \gamma_j^{\nu'(1-v, \lambda_1)+1} (t_j - t_{j-1})^{m+1-v} + \gamma_j^{\nu'(1-v, \lambda_1)+1} (t_j - t_{j-1})^{m+1} \right] \\
\leq c_1 \left[ N^{-r}\gamma_j^{\nu'(1-v, \lambda_1)+m} j^{\nu'(1-v, \lambda_1)+m-1+v} + N^{-r}\gamma_j^{\nu'(1-v, \lambda_1)+m} j^{\nu'(1-v, \lambda_1)+m-1} \right] \\
\leq c_2 \gamma_j,
\]
where
\[
\gamma_j = N^{-r}\gamma_j^{\nu'(1-v, \lambda_1)+m} j^{\nu'(1-v, \lambda_1)+m-1+v} \\
= N^{-m+1+v} \left( \frac{j}{N} \right) \gamma_j^{\nu'(1-v, \lambda_1)+m-1+v}.
\]
This together with (6.7) and Lemma 4.1 yields

\[
\left| \int_{t_{j-1}}^{t} K_\theta(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \\
\leq c \sup_{t_{j-1} \leq s \leq t_j} \left| (P_N^{(m-1)} v)(s) - v(s) \right| \left| K_\theta(t, s) \right| ds \\
\leq c_1 \gamma_j \left\{ \begin{array}{ll}
1 & \text{for } m < q(1 - \nu - \lambda_1) \\
1 + |\log t_j| & \text{for } m = q(1 - \nu - \lambda_1) \\
\frac{t_j}{t_j^{1-q(1-\nu-\lambda_1)+m}} & \text{for } m > q(1 - \nu - \lambda_1) \\
\end{array} \right. \\
\leq c_2 \left\{ \begin{array}{ll}
N^{-m-1+r} & \text{for } m \leq q(1 - \nu - \lambda_1) \\
N^{-2q(1-\nu-\lambda_1)} t_j^{2q(1-\nu-\lambda_1)-m-1+r} & \text{for } m > q(1 - \nu - \lambda_1) \\
\end{array} \right. \\
\leq c_3 E_N^{(m, \nu, \lambda_1, \varrho, r)}, \quad t_{j-1} \leq t \leq t_j, \quad 2 \leq j \leq N.
\]

In a similar way we get for any \( t \in [t_{j-1}, t_j] \) \((3 \leq j \leq N)\) that

\[
\left| \int_{t_{j-2}}^{t_{j-1}} K_\theta(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \leq c E_N^{(m, \nu, \lambda_1, \varrho, r)}.
\]

Combining these estimates and (6.7) with the equality

\[
\int_{t_{j-2}}^{t} K_\theta(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \\
= \left( \int_{t_{j-2}}^{t_{j-1}} + \int_{t_{j-1}}^{t} \right) K_\theta(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds,
\]

we obtain

\[
\left| \int_{t_{j-2}}^{t} K_\theta(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \leq c E_N^{(m, \nu, \lambda_1, \varrho, r)}, \quad (6.9)
\]

where \( t_{j-1} \leq t \leq t_j, \quad 2 \leq j \leq N. \)

Let us estimate \( S(t), \quad t_1 \leq t \leq b. \) Fix \( t, \) let \( k \in \{2, \ldots, N\} \) be such that \( t \in [t_{k-1}, t_k]. \) Then we can write

\[
S(t) = \sum_{j=1}^{k-2} \int_{t_{j-1}}^{t_j} K_\theta(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \\
+ \int_{t_{k-2}}^{t} K_\theta(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds. \quad (6.10)
\]
Due to (6.9) we have to estimate only the first term in the expression (6.10). To estimate it, in addition to the parameters \( \eta_1, \ldots, \eta_m \) (see (4.4)), we introduce a parameter \( \eta_{m+1} \in [0, 1] \) so that \( \eta_{m+1} \neq \eta_l, \ l = 1, \ldots, m \) (the choice of \( \eta_{m+1} \) in \([0, 1]\) is arbitrary but we assume that it is somehow fixed). Using \( \eta_1, \ldots, \eta_{m+1} \), we can define \( m + 1 \) distinct node points in every interval \([t_{j-1}, t_j]\) so that
due to (6.10) we have to estimate only the first term in the expression (6.10). To estimate it, in addition to the parameters \( \eta_1, \ldots, \eta_m \) (see (4.4)), we introduce a parameter \( \eta_{m+1} \in [0, 1] \) so that \( \eta_{m+1} \neq \eta_l, \ l = 1, \ldots, m \) (the choice of \( \eta_{m+1} \) in \([0, 1]\) is arbitrary but we assume that it is somehow fixed). Using \( \eta_1, \ldots, \eta_{m+1} \), we can define \( m + 1 \) distinct node points in every interval \([t_{j-1}, t_j]\): \( t_l = t_{j-1} + \eta_l(t_j - t_{j-1}), \ l = 1, \ldots, m + 1; \ j = 1, \ldots, N \).

Similarly to \( P_N^{(m-1)} \) let \( P_N^{(m)} \) be an interpolation operator that assigns to any continuous function \( z \in C[0, b] \) a piecewise polynomial function \( P_N^{(m)} z \) so that \( P_N^{(m)} z \) is on every interval \([t_{j-1}, t_j]\) \( (j = 1, \ldots, N) \) a polynomial of degree not exceeding \( m \) and \( (P_N^{(m)} z)(t_j) = z(t_j), \ l = 1, \ldots, m + 1; \ j = 1, \ldots, N \).

Since the quadrature approximation (6.1) is exact for all polynomials of degree not exceeding \( m \), we obtain that

\[
\int_{t_{j-1}}^{t_j} z(s) \, ds = \frac{t_j - t_{j-1}}{2} \sum_{l=1}^{m} \omega_l z(t_l), \quad j = 1, \ldots, N,
\]

where \( z \) is a polynomial of degree not exceeding \( m \). Therefore,

\[
\int_{t_{j-1}}^{t_j} [(P_N^{(m-1)} v)(s) - (P_N^{(m)} v)(s)] \, ds = 0, \quad j = 1, \ldots, N.
\]

Using this we can write

\[
\sum_{j=2}^{k-2} \int_{t_{j-1}}^{t_j} K_\varphi(t, s) \left[ (P_N^{(m-1)} v)(s) - v(s) \right] \, ds
\]

\[
= \sum_{j=2}^{k-2} \int_{t_{j-1}}^{t_{j+2}} \left[ K_\varphi(t, s) - K_\varphi(t, t_{j/2}) \right] \left[ (P_N^{(m-1)} v)(s) - v(s) \right] \, ds
\]

\[
+ \sum_{j=2}^{k-2} K_\varphi(t, t_{j/2}) \int_{t_{j-1}}^{t_j} \left[ (P_N^{(m)} v)(s) - v(s) \right] \, ds,
\]

with \( t_{j/2} = (t_{j-1} + t_j)/2, \ 2 \leq j \leq k - 2, \ 4 \leq k \leq N \).

Let us estimate the first term on the right-hand side of (6.11). Using (1.2)–(1.4), (3.1), and (5.2), we have for any \( s \in [t_{j-1}, t_j] \) \( (2 \leq j \leq k - 2) \) that

\[
K_\varphi(t, s) - K_\varphi(t, t_{j/2}) = (s - t_{j/2}) \frac{\partial}{\partial s} K_\varphi(t, s) \big|_{s=\xi}, \quad \xi \in (s, t_{j/2}),
\]

\[
\frac{\partial}{\partial s} K_\varphi(t, s) \big|_{s=\xi} = \frac{\partial}{\partial \gamma} K(\varphi(t, \gamma)) \big|_{\gamma=\varphi(\xi)} [\varphi'(\xi)]^2 + K(\varphi(t, \xi)) \varphi''(\xi)
\]

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This together with Lemma 4.1 yields
\[
\left. \frac{\partial}{\partial y} K_i(\varphi(t), y) \right|_{y=\xi(t)} (\varphi(t) - \varphi(\xi))^{-v} + vK_1(\varphi(t), \varphi(\xi))(\varphi(t) - \varphi(\xi))^{-v-1} + \frac{\partial}{\partial y} K_2(\varphi(t), y) \right|_{y=\xi(t)} [\varphi'(\xi)]^2 + [K_1(\varphi(t), \varphi(\xi))(\varphi(t) - \varphi(\xi))^{-v} + K_2(\varphi(t), \varphi(\xi))] \varphi''(\xi),
\]
\[
\left. \frac{\partial}{\partial s} K_\varphi(t,s) \right|_{s=\xi(t)} \leq c \left[ t_j^{(1-v-\lambda_1)+v-2}(t_k-1-t_j)^{-v} + t_j^{(1-v-\lambda_1)+v-1}(t_k-1-t_j)^{-v-1} + t_j^{(1-2\lambda_2)-2}(t_k-1-t_j)^2 \right]. \tag{6.13}
\]
 Further, we have
\[
t_k-1-t_j \geq (k-1-j)(t_j-t_{j-1}), \quad j = 2, \ldots, k-2,
\]
hence,
\[
(t_k-1-t_j)^{-z} \leq (k-1-j)^{-z}(t_j-t_{j-1})^{-z}, \quad z > 0, \ j = 2, \ldots, k-2. \tag{6.14}
\]
On the basis of (6.12)–(6.14) we get
\[
\int_{t_j}^{t_k} \left| K_\varphi(t,s) - K_\varphi(t,t_j/2) \right| ds \leq c \left[ t_j^{(1-v-\lambda_1)+v-2}(t_j-t_{j-1})^{2-v}(k-1-j)^{-v} + t_j^{(1-v-\lambda_1)+v-1}(t_j-t_{j-1})^{-v}(k-1-j)^{-v-1} + t_j^{(1-2\lambda_2)-2}(t_j-t_{j-1})^2 \right].
\]
This together with Lemma 4.1 yields
\[
\left| \sum_{j=2}^{k-2} \int_{t_{j-1}}^{t_j} \left[ K_\varphi(t,s) - K_\varphi(t,t_{j-1}/2) \right] \left[ (P_N^{(m-1)}v)(s) - v(s) \right] ds \right| \leq c \sum_{j=2}^{k-2} \left( \gamma_j^1 + \gamma_j^2 + \gamma_j^3 \right) \left\{ \begin{array}{ll} 1 & \text{for } m < \varrho(1-v-\lambda_1) \\
1 + |\log t_j| & \text{for } m = \varrho(1-v-\lambda_1) \\
\theta_j^{(1-v-\lambda_1)-m} & \text{for } m > \varrho(1-v-\lambda_1) \end{array} \right. \tag{6.15}
\]
where
\[
\begin{align*}
\gamma_j^1 &= t_j^{(1-v-\lambda_1)+v-2}(t_j-t_{j-1})^{m+2-v}(k-1-j)^{-v}, \\
\gamma_j^2 &= t_j^{(1-v-\lambda_1)+v-1}(t_j-t_{j-1})^{m+1-v}(k-1-j)^{-v-1}, \\
\gamma_j^3 &= t_j^{(1-2\lambda_2)-2}(t_j-t_{j-1})^{m+2}. 
\end{align*}
\]
It is easy to see that
\[
\begin{align*}
\gamma_j^1 & \leq c N^{-r(1-v-j_1) + m} \int (1-v-j_1) + m \cdot -m \cdot 2 + (k - j)^{-v}, \\
\gamma_j^2 & \leq c N^{-r(1-v-j_2) + m} \int (1-v-j_2) + m \cdot -m \cdot 1 + (k - j)^{-v - 1}, \\
\gamma_j^3 & \leq c N^{-r(1-v-j_3) + m} \int (1-v-j_3) + m \cdot -m \cdot 2,
\end{align*}
\] (6.16)
where \(2 \leq j \leq k - 2\).

If \(m < q(1 - v - j_1)\), then \(r[q(1 - v - j_1) + m] - m - 1 + v > 0\) and (6.16) implies
\[
\begin{align*}
\gamma_j^1 & \leq c N^{-m-1+v} \left( \frac{j}{N} \right)^{r[1-v-j_1] + m} \cdot -m \cdot 1 + (k - j)^{-v}, \\
& \leq c N^{-m-1+v} j^{-1}(k - j)^{-v}, \\
\gamma_j^2 & \leq c N^{-m-1+v}(k - j)^{-v - 1}, \\
\gamma_j^3 & \leq c N^{-m-1+v} j^{-1-v}.
\end{align*}
\]
This together with (6.15) yields
\[
\begin{align*}
\left| \sum_{j=2}^{k-2} \int_{j-1}^{j} \left[ K_v(t, s) - K_v(t, j/2) \right] \left( \left( P_N^{(m-1)} v \right)(s) - v(s) \right) ds \right| \\
& \leq c N^{-m-1+v} \sum_{j=2}^{k-2} \left[ j^{-1}(k - j)^{-v} + (k - j)^{-v - 1} + j^{-1-v} \right] \\
& \leq c N^{-m-1+v}.
\end{align*}
\]
In the case \(m = q(1 - v - j_1)\) it follows from (6.16) that
\[
\begin{align*}
(\gamma_j^1 + \gamma_j^2 + \gamma_j^3) \left( 1 + \left| \log \frac{j}{N} \right| \right) \\
& \leq c N^{-m-1+v} \left( \frac{j}{N} \right)^{(r-1)m + rm - 1 + v} \left( 1 + \left| \log \frac{j}{N} \right| \right) j^{-1}(k - j)^{-v} \\
& + c_1 N^{-m-1+v} \left( \frac{j}{N} \right)^{(r-1)m + rm - 1 + v} (1 + \left| \log \frac{j}{N} \right|) (k - j)^{-v - 1} \\
& + c_2 N^{-m-1+v} \left( \frac{j}{N} \right)^{(r-1)m + rm - 1 + v} (1 + \left| \log \frac{j}{N} \right|) j^{-1-v} \\
& \leq c N^{-m-1+v} \left[ j^{-1}(k - j)^{-v} + (k - j)^{-v - 1} + j^{-1-v} \right],
\end{align*}
\]
and, due to (6.15), we obtain

\[
\left| \sum_{j=2}^{k-2} \int_{t_{j-1}}^{t_j} \left[ K_{\varphi}(t, s) - K_{\varphi}(t, t_{j/2}) \right] \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \leq c N^{-m+1}. 
\]

If \( m > \varrho(1 - v - \lambda_1) \), then using (6.15) and (6.16), we get

\[
\left| \sum_{j=2}^{k-2} \int_{t_{j-1}}^{t_j} \left[ K_{\varphi}(t, s) - K_{\varphi}(t, t_{j/2}) \right] \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \leq c N^{-2\varrho(1-v-\lambda_1)} \sum_{j=2}^{k-2} 2^{2\varrho(1-v-\lambda_1)-m-1} \times \left[ j^{-1}(k - 1 - j)^{-r} + (k - 1 - j)^{-r-1} + j^{-1-r} \right]
\]

\[
\leq c_1 \begin{cases} 
N^{r-2\varrho(1-v-\lambda_1)} & \text{for } 1 \leq r < \frac{m + 1 - v}{2\varrho(1 - v - \lambda_1)}, \\
N^{-m+1} & \text{for } r \geq \frac{m + 1 - v}{2\varrho(1 - v - \lambda_1)}. 
\end{cases}
\]

Summarizing these cases we have shown that

\[
\left| \sum_{j=2}^{k-2} \int_{t_{j-1}}^{t_j} \left[ K_{\varphi}(t, s) - K_{\varphi}(t, t_{j/2}) \right] \left[ (P_N^{(m-1)} v)(s) - v(s) \right] ds \right| \leq c \mathcal{E}_N^{(m,v,\lambda_1,\varrho,r)},
\]

(6.17)

with a constant \( c > 0 \) that is independent of \( N \).

It remains to estimate the second term on the right-hand side of (6.11). Since \( v \in C^{m+1,\tilde{\vartheta}}(0, b), \tilde{\vartheta} = 1 - \varrho(1 - v - \lambda_1) \), we get from Lemma 4.1 (with \( m + 1 \) instead of \( m \)) the estimate

\[
\sup_{t_{j-1} \leq t \leq t_j} \left| (P_N^{(m)} v)(s) - v(s) \right| 
\leq c(t_j - t_{j-1})^{m+1} \begin{cases} 
1 & \text{for } m + 1 < \varrho(1 - v - \lambda_1), \\
1 + |\log t_j| & \text{for } m + 1 = \varrho(1 - v - \lambda_1), \\
t_j^{r(1-v-\lambda_1)-m-1} & \text{for } m + 1 > \varrho(1 - v - \lambda_1). 
\end{cases}
\]

(6.18)

Since \( t_{j/2} < t_j < t \), it follows from (1.2)–(1.4), (5.2), and (6.14) that

\[
|K_{\varphi}(t, t_{j/2})| \leq c \left[ t_j^{r(1-v-\lambda_1)+r-1}(t_j - t_{j-1})^{-r}(k - 1 - j)^{-r} + t_j^{r(1-\lambda_2)-1} \right].
\]
This together with (6.18) yields

\[
\left| \sum_{j=2}^{k-2} K_\varphi(t, t_{j/2}) \int_{t_{j-1}}^{t_j} [ (P_N^{(m)} v)(s) - v(s) ] \, ds \right|
\]

\[
\leq c \sum_{j=2}^{k-2} (\tilde{\gamma}^1_j + \tilde{\gamma}^2_j) \begin{cases} 
1 & \text{for } m + 1 < \varrho(1 - \nu - \lambda_1), \\
1 + |\log t_j| & \text{for } m + 1 = \varrho(1 - \nu - \lambda_1), \\
\log (t_j - t_{j-1}) & \text{for } m + 1 > \varrho(1 - \nu - \lambda_1),
\end{cases}
\]

where

\[
\tilde{\gamma}^1_j = t_j^{\varrho(1 - \nu - \lambda_1) + \nu - 1} (t_j - t_{j-1})^{m+2-v} (k - 1 - j)^{-\nu},
\]

\[
\tilde{\gamma}^2_j = t_j^{\varrho(1 - \lambda_2) - 1} (t_j - t_{j-1})^{m+2}.
\]

We observe that

\[
\tilde{\gamma}^1_j \leq c N^{-m+2+\nu} \left( \frac{j}{N} \right)^{\varrho(1 - \nu - \lambda_1) + m + 1 - m - 2 + \nu} (k - 1 - j)^{-\nu},
\]

\[
\tilde{\gamma}^2_j \leq c N^{-m+2} \left( \frac{j}{N} \right)^{\varrho(1 - \nu - \lambda_1) + m + 1 - m - 2},
\]

where \(2 \leq j \leq k - 2\).

If \(m + 1 < \varrho(1 - \nu - \lambda_1)\), then \(\varrho(1 - \nu - \lambda_1) + m + 1 - m - 2 > 0\), hence

\[
\tilde{\gamma}^1_j \leq c N^{-m+2+\nu} (k - 1 - j)^{-\nu}, \quad \tilde{\gamma}^2_j \leq c N^{-m+2},
\]

\[
\tilde{\gamma}^1_j \left( 1 + |\log \frac{j}{N}| \right) \leq c N^{-m+2+\nu} (k - 1 - j)^{-\nu}, \quad \tilde{\gamma}^2_j \left( 1 + |\log \frac{j}{N}| \right) \leq c N^{-m+2},
\]

and it follows from (6.19) that

\[
\left| \sum_{j=2}^{k-2} K_\varphi(t, t_{j/2}) \int_{t_{j-1}}^{t_j} [ (P_N^{(m)} v)(s) - v(s) ] \, ds \right|
\]

\[
\leq c \left[ N^{-m+2+\nu} \sum_{j=2}^{k-2} (k - 1 - j)^{-\nu} + N^{-m+1} \right] \leq c_1 N^{-m+1}. \quad (6.21)
\]

If \(m + 1 > \varrho(1 - \nu - \lambda_1)\), then (6.19) and (6.20) imply

\[
\left| \sum_{j=2}^{k-2} K_\varphi(t, t_{j/2}) \int_{t_{j-1}}^{t_j} [ (P_N^{(m)} v)(s) - v(s) ] \, ds \right|
\]
\[
\begin{align*}
&\leq c \left[ N^{-2q(1-v-l_1)} \sum_{j=2}^{k-2} j^{2q(1-v-l_1)-m-1+r-j} (k-1-j)^{-r} \\
&\quad + N^{-2q(1-v-l_1)} \sum_{j=2}^{k-2} j^{2q(1-v-l_1)-m-2} \right] \\
&\leq cN^{-2q(1-v-l_1)} \begin{cases} 
1 & \text{for } 1 \leq r < \frac{m+1-v}{2q(1-v-l_1)}, \\
N^{-m-1+r} & \text{for } r \geq \frac{m+1-v}{2q(1-v-l_1)}, \quad r \geq 1.
\end{cases}
\end{align*}
\]

Summarizing these cases we obtain from (6.19), (6.21), and (6.22) that
\[
\left| \sum_{j=2}^{k-2} K_\nu(t, t_j/2) \int_{t_j-1}^{t_j} \left[ (P_N^{(m)}(v)(s) - v(s) \right) ds \right| \leq cE_N^{(m,r,l_1,q,r)},
\]
with a constant \(c > 0\) that is independent of \(N\). It follows from (6.8)–(6.11), (6.17), and (6.23) the estimate
\[
\sup_{t_1 \leq t \leq t_2} |S(t)| \leq cE_N^{(m,r,l_1,q,r)},
\]
where \(c > 0\) is independent of \(N\). This together with (6.4), (6.5), and (6.6) yields (6.2).

## 7 NUMERICAL EXAMPLES

Let us consider the following equation:
\[
\begin{align*}
u(x) &= \int_0^x \left( (x-y)^{\gamma-\gamma_1} + y^{\gamma_2} \right) u(y) \, dy + f(x), \quad 0 \leq x \leq 1,
\end{align*}
\]
where \(0 < \gamma < 1, \lambda_1 < 1, \gamma + \lambda_1 < 1, \lambda_2 < \gamma + \lambda_1\). The forcing function \(f\) is selected so that \(u(x) = x^{1-\gamma_1} - \gamma x^\beta + 1, \quad \beta \geq 1\), is the exact solution to (7.1). Actually, this is a problem of the form (1.1)–(1.3) where \(b = 1\), \(K_1(x, y) = y^{-\lambda_1}, \quad K_2(x, y) = y^{-\lambda_2}\),
\[
f(x) = \left( 1 - \frac{\Gamma(1-\gamma)\Gamma(1-\lambda_1)}{\Gamma(2-\gamma-\lambda_1)} \right) x^{1-\gamma_1} - \gamma x^\beta + 1 \\
- \frac{\Gamma(1-\gamma)\Gamma(2-\gamma-2\lambda_1)}{\Gamma(3-2\gamma-2\lambda_1)} x^{2(1-\gamma_1)} + v \frac{\Gamma(1-\gamma)\Gamma(1+\beta-\lambda_1)}{\Gamma(2+\beta-\gamma-\lambda_1)} x^{1+\beta-\gamma_1} \\
- \frac{x^{2-\gamma_1-\lambda_2}}{2-\gamma-\lambda_1-\lambda_2} + v \frac{x^{1+\beta-\lambda_2}}{1+\beta-\lambda_2} - \frac{x^{1-\lambda_2}}{1-\lambda_2}, \quad 0 \leq x \leq 1,
\]
\[
\Gamma(t) = \int_0^\infty e^{-t} s^{t-1} ds, \quad t > 0.
\]
It is easy to check that in this case \( f \in C^{m+r+\lambda_1}(0,1) \) for arbitrary \( m \in \mathbb{N} \).

Equation (7.1) was solved numerically by method (5.3)–(5.5) for \( \beta = 1.5 \) and

\[
m = 3, \quad v = \frac{3}{10}, \quad \lambda_1 = \frac{2}{10}, \quad \lambda_2 = \frac{49}{100},
\]

\[
\eta_1 = \frac{5 - \sqrt{15}}{10}, \quad \eta_2 = \frac{1}{2}, \quad \eta_3 = \frac{5 + \sqrt{15}}{10}.
\]

Here \( \eta_1, \eta_2, \eta_3 \) are the node points of the Gauss–Legendre quadrature rule (6.1) by \( m = 3 \). This formula is exact for all polynomials of degree not exceeding \( 2m - 1 = 5 \).

In Tables 1–4 some results for different values of the parameters \( N, \varrho, \) and \( r \) are presented. The quantities \( \sigma_N^{(q,r)} \) in Tables 1 and 2 are approximate values of the norm \( \| u_N - u \|_\infty \), calculated as follows:

\[
\sigma_N^{(q,r)} = \max_{l=0, \ldots, 10} \max_{j=1, \ldots, N} |u_N((\tau_{jl}^{(r)})(\varphi)) - u((\tau_{jl}^{(r)})(\varphi))|,
\]

where

\[
\tau_{jl}^{(r)} = t_{j-1} + l(t_j - t_{j-1})/10, \quad l = 0, \ldots, 10; \quad j = 1, \ldots, N,
\]

with the grid points \( t_j = t_j^{(r)} \), defined by formula (4.2) for \( b = 1 \). Tables 3 and 4 show the dependence of

\[
\tilde{\sigma}_N^{(q,r)} = \max_{l=1, \ldots, m} \max_{j=1, \ldots, N} |u_N(t_j^{(r)}) - u(t_j^{(r)})| = \max_{l=1, \ldots, m} \max_{j=1, \ldots, N} |v_N(t_j^{(r)}) - u(t_j^{(r)})|,
\]

**TABLE 1** Maximal errors and the corresponding ratios for \( r = 1 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \sigma_N^{(1,1)} )</th>
<th>( \tilde{\sigma}_N^{(1,1)} )</th>
<th>( \sigma_N^{(5,2,1)} )</th>
<th>( \tilde{\sigma}_N^{(5,2,1)} )</th>
<th>( \sigma_N^{(3,7,1)} )</th>
<th>( \tilde{\sigma}_N^{(3,7,1)} )</th>
<th>( \sigma_N^{(6,1,1)} )</th>
<th>( \tilde{\sigma}_N^{(6,1,1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.15</td>
<td>2.83</td>
<td>3.04</td>
<td>4.19</td>
<td>7.60</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.8 E-1</td>
<td>1.3 E-4</td>
<td>7.7 E-5</td>
<td>1.4 E-5</td>
<td>3.9 E-5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.12</td>
<td>2.83</td>
<td>3.04</td>
<td>3.61</td>
<td>7.86</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>2.10</td>
<td>2.83</td>
<td>3.04</td>
<td>3.61</td>
<td>7.97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>8.6 E-2</td>
<td>4.4 E-5</td>
<td>2.6 E-5</td>
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<td></td>
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</tr>
<tr>
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<td>2.83</td>
<td>3.04</td>
<td>3.61</td>
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<tr>
<td>128</td>
<td>2.08</td>
<td>2.83</td>
<td>3.03</td>
<td>3.61</td>
<td>8.00</td>
<td></td>
<td></td>
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<tr>
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<td>1.6 E-5</td>
<td>8.4 E-6</td>
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<td>6.1 E-7</td>
<td></td>
<td></td>
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</tr>
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<td>2.83</td>
<td>3.03</td>
<td>3.61</td>
<td>8.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>256</td>
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<td>2.83</td>
<td>3.03</td>
<td>3.61</td>
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<td></td>
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<tr>
<td></td>
<td>2.0 E-2</td>
<td>5.6 E-6</td>
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<td>2.9 E-7</td>
<td>7.7 E-8</td>
<td></td>
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<td>2.83</td>
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<tr>
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<td>9.5 E-3</td>
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<td>9.2 E-7</td>
<td>8.1 E-8</td>
<td>9.6 E-9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.41</td>
<td>2.83</td>
<td>3.03</td>
<td>3.61</td>
<td>8.00</td>
<td></td>
<td></td>
<td></td>
</tr>
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</table>
TABLE 2 Maximal errors and the corresponding ratios for \( r > 1 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \sigma_N^{(1,2)} )</th>
<th>( \sigma_N^{(1,2.5)} )</th>
<th>( \sigma_N^{(3,1.05)} )</th>
<th>( \sigma_N^{(3,2)} )</th>
<th>( \sigma_N^{(4,5,2)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>3.12</td>
<td>2.40</td>
<td>2.98</td>
<td>7.79</td>
<td>7.25</td>
</tr>
<tr>
<td>64</td>
<td>3.2 E-3</td>
<td>2.38</td>
<td>2.98</td>
<td>8.02</td>
<td>7.90</td>
</tr>
<tr>
<td>128</td>
<td>1.6 E-3</td>
<td>1.1 E-3</td>
<td>3.3 E-5</td>
<td>3.9 E-7</td>
<td>1.7 E-6</td>
</tr>
<tr>
<td>256</td>
<td>2.01</td>
<td>2.38</td>
<td>2.98</td>
<td>8.03</td>
<td>7.98</td>
</tr>
<tr>
<td>512</td>
<td>2.00</td>
<td>2.38</td>
<td>2.98</td>
<td>8.03</td>
<td>8.01</td>
</tr>
</tbody>
</table>

On the parameters \( N, \varrho, \) and \( r \) (see (6.2)). The ratios

\[
\delta_N^{(p,r)} = \sigma_N^{(p,r)}/\sigma_N, \quad \tilde{\delta}_N^{(p,r)} = \tilde{\sigma}_N^{(p,r)}/\tilde{\sigma}_N,
\]

characterizing the observed convergence rate, are also presented.

From Theorem 5.4 it follows that for sufficiently large \( N \),

\[
\sigma_N^{(p,r)} \approx \| u_N - u \|_\infty \leq \begin{cases} N^{-3} & \text{if } \varrho > 6, \ r \geq 1, \\
N^{-3} & \text{if } \varrho = 6, \ r > 1, \\
N^{-3} & \text{if } \varrho < 6, \ r \geq 6/\varrho, \\
N^{-\varrho/2} & \text{if } \varrho < 6, \ 1 \leq r < 6/\varrho. \end{cases} \tag{7.2}
\]

TABLE 3 Errors at the special points and the corresponding ratios for \( r = 1 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \tilde{\sigma}_N^{(1,1)} )</th>
<th>( \tilde{\sigma}_N^{(3,1)} )</th>
<th>( \tilde{\sigma}_N^{(5,2,1)} )</th>
<th>( \tilde{\sigma}_N^{(5,7,1)} )</th>
<th>( \tilde{\sigma}_N^{(6,1,1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.13</td>
<td>7.61</td>
<td>9.12</td>
<td>12.63</td>
<td>12.17</td>
</tr>
<tr>
<td>64</td>
<td>2.11</td>
<td>7.84</td>
<td>9.28</td>
<td>12.92</td>
<td>12.52</td>
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<tr>
<td>128</td>
<td>2.09</td>
<td>7.99</td>
<td>9.38</td>
<td>13.07</td>
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<tr>
<td>256</td>
<td>2.07</td>
<td>8.03</td>
<td>9.00</td>
<td>13.48</td>
<td>13.74</td>
</tr>
<tr>
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<td>8.00</td>
<td>9.19</td>
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</tbody>
</table>
TABLE 4 Errors at the special points and the corresponding ratios for \( r > 1 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \frac{\delta_N^{(1,2)}}{\delta_N^{(1,2)}} )</th>
<th>( \frac{\delta_N^{(1,3)}}{\delta_N^{(1,3)}} )</th>
<th>( \frac{\delta_N^{(3,1,0)}}{\delta_N^{(3,1,0)}} )</th>
<th>( \frac{\delta_N^{(5,2)}}{\delta_N^{(5,2)}} )</th>
<th>( \frac{\delta_N^{(4,5,2)}}{\delta_N^{(4,5,2)}} )</th>
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</thead>
<tbody>
<tr>
<td>32</td>
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<td>8.66</td>
<td>12.67</td>
<td>11.49</td>
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<td>8.85</td>
<td>12.72</td>
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<td>6.7 E-8</td>
<td>9.8 E-8</td>
<td>5.3 E-7</td>
</tr>
<tr>
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<td>5.82</td>
<td>8.97</td>
<td>12.89</td>
<td>12.63</td>
</tr>
<tr>
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<td>2.4 E-5</td>
<td>7.5 E-9</td>
<td>7.6 E-9</td>
<td>4.2 E-8</td>
</tr>
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<td>9.01</td>
<td>13.11</td>
<td>12.94</td>
</tr>
<tr>
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<td>8.88</td>
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</tr>
</tbody>
</table>

Due to (7.2), the ratios \( \delta_N^{(1,1)} \), \( \delta_N^{(3,1)} \), \( \delta_N^{(5,2,1)} \), \( \delta_N^{(3,7,1)} \), and \( \delta_N^{(6,1,1)} \) in Table 1 ought to be approximately 1.41, 2.83, 3.03, 3.61, and 8.00, respectively. These values of \( \delta_N^{(6,7)} \) are given in the last row of Tables 1 and 2.

In a similar way we obtain by Theorem 6.1 (see (6.3)) that \( \delta_N^{(1,1)} \), \( \delta_N^{(5,1)} \), \( \delta_N^{(5,2,1)} \), \( \delta_N^{(3,7,1)} \), and \( \delta_N^{(6,1,1)} \) in Table 3 ought to be approximately 2.00, 2.38, 2.98, 8.00, and 8.00, respectively. These values of \( \delta_N^{(6,7)} \) are given in the last row of Tables 3 and 4.

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REFERENCES


